Shift and delta operator realisations for digital controllers with finite word length considerations

J.Wu, S.Chen, G.Li, R.H.Istepanian and J.Chu

Abstract: The implementation issues of digital controllers with finite word length (FWL) considerations are addressed. Both the shift and delta operator parameterisations of a general controller structure are considered. A unified formulation is adopted to derive a computationally tractable stability related measure that describes FWL closed-loop stability characteristics of different controller realisations. Within a given operator parameterisation, the optimal FWL controller realisation, which maximises the proposed stability related measure, is the solution of a nonlinear optimisation problem. The relationship between the z-operator and δ-operator controller parameterisations is analysed, and it is shown that the δ parameterisation has a better FWL closed-loop stability margin than the z-domain approach under a mild condition. A design example is included to verify the theoretical analysis and to illustrate the proposed optimisation procedure.

1 Introduction

Modern controllers are typically implemented digitally, and it is well known that a designed stable control system may achieve a lower than predicted performance, or even become unstable when the control law is implemented with a finite-precision device due to FWL effects. For many industrial and mass-market consumer applications, fixed-point implementations are more desirable for reasons of cost, simplicity, speed, memory space and power consumption. With a fixed-point processor, however, the detrimental FWL effects are markedly increased due to reduced precision. The FWL effects on the closed-loop stability depend on the controller realisation structure. This property can be utilized to 'select' controller realisation in order to improve the 'robustness' of closed-loop stability under controller parameter perturbations. Currently, two approaches exist for determining the optimal controller realisations under different criteria, namely pole sensitivity measures [1–4] and complex stability radius measures [5, 6].

In the first approach, the pole sensitivity measures based on an l_2 norm [2] and an l_1 norm [3] are used to quantify the FWL effects on closed-loop stability. This approach leads to a nonlinear and non-smooth optimisation problem in finding an optimal FWL controller realisation. The need to solve for such a non-convex and non-smooth optimisation problem had been seen as a disadvantage, as conventional optimisation algorithms [7, 8], which are better known to the control community, may not guarantee to find a true optimal realisation. However, the efficient global optimisation techniques to tackle this kind of difficult optimisation problem [9–14] are now widely available. More recently, Fialho and Georgiou [6] used the complex stability radius measure to formulate an optimal FWL controller realisation problem that can be represented as a special H_∞ norm minimisation problem, and solved with the method of linear matrix inequality [15, 16]. In this second approach, the FWL perturbations are assumed to be complex-valued. Although this assumption is somewhat artificial, the approach based on the complex stability radius measure has certain attractive features and requires further investigation.

Most studies on the FWL stability issues only consider the closed-loop systems with output feedback (OF) controllers. It is well known that there exists another class of controllers, namely observer-based (OB) controllers [17, 18]. Because state-space methods and observer theory are combined to form a direct multi-variable approach to linear control system synthesis [18], the design of OB controllers is more transparent and simpler than the design of OF controllers. Li and Gevers [19] have studied the sensitivity and the roundoff noise gain of the closed-loop system transfer function with an FWL implemented full-order OB controller. A recent study [20] has investigated the effects of FWL implementation on the closed-loop stability for full-order OB controllers. The first contribution of this paper is to develop a new framework of optimal FWL controller realisations for the generic digital controller structure that includes all the OF and OB controllers. A computationally tractable stability related measure is employed for the unified controller structure, using the well-tested pole sensitivity measure with the l_1 norm [3].

In most of the above-mentioned studies, digital controller structures are described and realised with the usual shift

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J. Wu and J. Chu are with the National Laboratory of Industrial Control Technology, Institute of Advanced Process Control, Zhejiang University, Hangzhou 310027, P. R. China

S. Chen is with the Department of Electronics and Computer Science, University of Southampton, Highfield, Southampton SO17 1BJ, UK
E-mail: sqc@ecs.soton.ac.uk

G. Li is with the School of of Electrical and Electronic Engineering, Nanyang Technological University, Singapore

R.H. Istepanian is with the Department of Electronic and Computer Engineering, Brunel University, Uxbridge UB8 3PH, UK

operator z. A discrete-time system can also be described and realised with a different operator, called the delta operator δ [21]. Two major advantages are known for the use of δ-operator parameterisation: a theoretically unified formulation of continuous-time and discrete-time systems, and better numerical properties in FWL implementations [1]. The benefits of using the δ-operator as opposed to the shift operator in signal processing and control applications have been investigated [22–25]. In particular, a recent work has addressed the FWL closed-loop stability issues of OF controller structures using the δ-operator formulation [26]. The second new contribution of this paper is to adopt a unified formulation to include both the z and δ-operator parameterisations of the generic finite-precision controller structure, and to analyse the underlying relationship between these two controller parameterisations.

2 Notations, definitions and problem formulation

Let $\mathcal{R}$ denote the field of real numbers and C the field of complex numbers. For a complex-valued matrix $\mathbf{U} \in C^{p \times q}$ with elements u_{ij} we define the following matrix norm:

$$\|\mathbf{U}\|_S \triangleq \sum_{i=1}^{p} \sum_{j=1}^{q} |u_{ij}| \tag{1}$$

Let Vec($\cdot$) be the column stacking operator such that Vec($\mathbf{U}$) is a qp-dimensional vector. As usual, $\mathbf{U}^T$ is the transposed matrix of $\mathbf{U}$, $\mathbf{U}^H$ is the Hermitian adjoint matrix of $\mathbf{U}$, and $\mathbf{U}^*$ is conjugate to $\mathbf{U}$. For a squared real-valued matrix $\mathbf{M} \in \mathcal{R}^{p \times p}$, let $\{\lambda_i(\mathbf{M}),\ 1 \le i \le p\}$ denote its eigenvalues. For diagonalisable $\mathbf{M}$, let $\mathbf{x}_i(\mathbf{M})$ be the right eigenvector corresponding to $\lambda_i(\mathbf{M})$, that is

$$\mathbf{M}\mathbf{x}_i(\mathbf{M}) = \lambda_i(\mathbf{M})\mathbf{x}_i(\mathbf{M}) \tag{2}$$

Since $\mathbf{M}$ is diagonalisable, the matrix

$$\mathbf{M}_x \triangleq \begin{bmatrix} \mathbf{x}_1(\mathbf{M}) & \cdots & \mathbf{x}_p(\mathbf{M}) \end{bmatrix} \tag{3}$$

is invertible. Define:

$$\mathbf{M}_y = \begin{bmatrix} \mathbf{y}_1(\mathbf{M}) & \cdots & \mathbf{y}_p(\mathbf{M}) \end{bmatrix} \triangleq \mathbf{M}_x^{-H} \tag{4}$$

$\mathbf{y}_i(\mathbf{M})$ is called the reciprocal left eigenvector corresponding to $\mathbf{x}_i(\mathbf{M})$ for the reason shown in the following lemma.

Lemma 1: $\mathbf{y}_i^H(\mathbf{M})\mathbf{M} = \lambda_i(\mathbf{M})\mathbf{y}_i^H(\mathbf{M}), \forall i$.

Proof: Denote

$$\Sigma \triangleq \begin{bmatrix} \lambda_i(\mathbf{M}) & & \\ & \ddots & \\ & & \lambda_p(\mathbf{M}) \end{bmatrix} \tag{5}$$

Clearly, $\mathbf{M}\mathbf{M}_x = \mathbf{M}_x\Sigma$. It then follows from $\mathbf{M}_x\mathbf{M}_y^H = \mathbf{M}_y^H\mathbf{M}_x = \mathbf{I}$, the identity matrix, that $\mathbf{M}_y^H\mathbf{M} = \Sigma\mathbf{M}_y^H$, which leads to lemma 1.

A discrete-time system can be described using either the usual z-operator or the so-called δ-operator. The latter is defined as [21]:

$$\delta \triangleq \frac{z-1}{h} \tag{6}$$

where h is a positive real constant [Note 1]. Let the state-space representation of a discrete-time system using the z-operator be

$$\begin{cases} z\mathbf{x}(k) = \mathbf{A}_z\mathbf{x}(k) + \mathbf{B}_z\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}_z\mathbf{x}(k) + \mathbf{D}_z\mathbf{u}(k) \end{cases} \tag{7}$$

where all the matrices and vectors are real-valued and are assumed to have proper dimensions, and $z\mathbf{x}(k) = \mathbf{x}(k+1)$, as z is the forward shift operator. We can describe the same discrete-time system by

$$\begin{cases} \delta\mathbf{x}(k) = \mathbf{A}_\delta\mathbf{x}(k) + \mathbf{B}_\delta\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}_\delta\mathbf{x}(k) + \mathbf{D}_\delta\mathbf{u}(k) \end{cases} \tag{8}$$

using the δ-operator [27, 28], where

$$\mathbf{A}_\delta = \frac{\mathbf{A}_z - \mathbf{I}}{h}, \ \mathbf{B}_\delta = \frac{\mathbf{B}_z}{h}, \ \mathbf{C}_\delta = \mathbf{C}_z, \ \mathbf{D}_\delta = \mathbf{D}_z \tag{9}$$

with $\mathbf{I}$ denoting the identity matrix of appropriate dimension. Obviously, eqns 7 and 8 are two equivalent representations of the same system. The following lemma relates the eigenvalues and eigenvectors of $\mathbf{A}_z$ to those of $\mathbf{A}_\delta$.

Lemma 2: With a proper index order, $\{\lambda_i(\mathbf{A}_z)\}$ and $\{\lambda_i(\mathbf{A}_\delta)\}$ can be one-to-one mapped with

$$\lambda_i(\mathbf{A}_z) = 1 + h\lambda_i(\mathbf{A}_\delta), \forall i \tag{10}$$

Let $\lambda_i(\mathbf{A}_\delta)$ and $\lambda_i(\mathbf{A}_z)$ be related by eqn. 10. Then they have the same eigenvector set.

Proof: Let $\mathbf{x}_i(\mathbf{A}_z)$ be an eigenvector corresponding to $\lambda_i(\mathbf{A}_z)$. It follows from eqn 9 that $\lambda_i(\mathbf{A}_z)\mathbf{x}_i(\mathbf{A}_z) = \mathbf{A}_z\mathbf{x}_i(\mathbf{A}_z) = h\mathbf{A}_\delta\mathbf{x}_i(\mathbf{A}_z) + \mathbf{x}_i(\mathbf{A}_z)$, which means that

$$\frac{\lambda_i(\mathbf{A}_z) - 1}{h}\mathbf{x}_i(\mathbf{A}_z) = \mathbf{A}_\delta\mathbf{x}_i(\mathbf{A}_z) \tag{11}$$

This, by definition, implies that $(\lambda_i(\mathbf{A}_z) - 1)/h$ is an eigenvalue of $\mathbf{A}_\delta$, denoted as $\lambda_i(\mathbf{A}_\delta)$, and $\mathbf{x}_i(\mathbf{A}_z)$ is also an eigenvector of $\mathbf{A}_\delta$, corresponding to $\lambda_i(\mathbf{A}_\delta)$. Using the same procedure, one can show that if $\mathbf{x}_i(\mathbf{A}_\delta)$ is an eigenvector of $\lambda_i(\mathbf{A}_\delta)$, it is also an eigenvector related to an eigenvalue of $\mathbf{A}_z$ given by eqn 10. This completes the proof.

It is well known that the discrete-time system ($\mathbf{A}_z$, $\mathbf{B}_z$, $\mathbf{C}_z$, $\mathbf{D}_z$) is stable if and only if

$$|\lambda_i(\mathbf{A}_z)| < 1, \ \forall i \tag{12}$$

From lemma 2, we have the stability condition for the same system described using the δ-operator.

Lemma 3: The discrete-time system ($\mathbf{A}_\delta$, $\mathbf{B}_\delta$, $\mathbf{C}_\delta$, $\mathbf{D}_\delta$) is stable if and only if

$$\left|\lambda_i(\mathbf{A}_\delta) + \frac{1}{h}\right| < \frac{1}{h}, \ \forall i \tag{13}$$

For notational conciseness, we introduce a 'generalised' operator ρ for the discrete-time systems. It is understood that $\rho = z$ or δ, depending on which operator is actually used. The two state-space representations (eqns 7 and 8) can then be unified as

$$\begin{cases} \rho\mathbf{x}(k) = \mathbf{A}_\rho\mathbf{x}(k) + \mathbf{B}_\rho\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}_\rho\mathbf{x}(k) + \mathbf{D}_\rho\mathbf{u}(k) \end{cases} \tag{14}$$

The use of this notation will avoid repeated derivations for the two operators in the following discussion.

[Note 1] In [21], h is limited to the sampling period. This constraint is removed in [24].

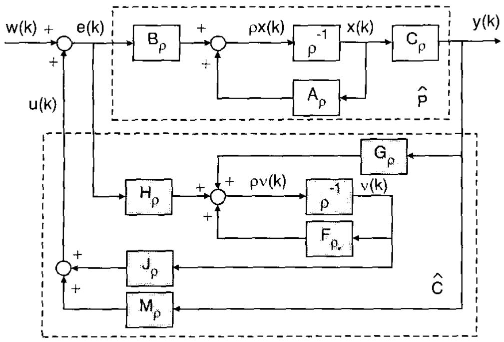

Fig. 1 *Discrete-time closed-loop system with generic controller*

Consider the discrete-time, closed-loop control system depicted in Fig. 1, where the linear time-invariant plant $\hat{P}$ has a state-space representation

$$\begin{cases} \rho \mathbf{x}(k) = \mathbf{A}_\rho \mathbf{x}(k) + \mathbf{B}_\rho \mathbf{e}(k) \\ \mathbf{y}(k) = \mathbf{C}_\rho \mathbf{x}(k) \end{cases} \tag{15}$$

which is assumed to be strictly proper, completely state controllable, and completely state observable, with $\mathbf{A}_\rho \in \mathcal{R}^{n\times n}$, $\mathbf{B}_\rho \in \mathcal{R}^{n\times p}$ and $\mathbf{C}_\rho \in \mathcal{R}^{q\times n}$, and the digital stabilising controller $\hat{C}$ is described by the state-space representation

$$\begin{cases} \rho \mathbf{v}(k) = \mathbf{F}_\rho \mathbf{v}(k) + \mathbf{G}_\rho \mathbf{y}(k) + \mathbf{H}_\rho \mathbf{e}(k) \\ \mathbf{u}(k) = \mathbf{J}_\rho \mathbf{v}(k) + \mathbf{M}_\rho \mathbf{y}(k) \end{cases} \tag{16}$$

where $\mathbf{F}_\rho \in \mathcal{R}^{m\times m}$, $\mathbf{G}_\rho \in \mathcal{R}^{m\times q}$, $\mathbf{J}_\rho \in \mathcal{R}^{p\times m}$, $\mathbf{M}_\rho \in \mathcal{R}^{p\times q}$ and $\mathbf{H}_\rho \in \mathcal{R}^{m\times p}$. The controller depicted in Fig. 1 is generic and includes all the OF and OB controllers: $\hat{C}$ is an OF controller when $\mathbf{H}_\rho = 0$; a full-order OB controller when $\mathbf{F}_\rho = \mathbf{A}_\rho - \mathbf{G}_\rho \mathbf{C}_\rho$, $\mathbf{M}_\rho = 0$ and $\mathbf{H}_\rho = \mathbf{B}_\rho$: a reduced-order OB controller, otherwise [17, 18].

It is a basic property of linear system theory that the state-space realisation ($\mathbf{F}_\rho$, $\mathbf{G}_\rho$, $\mathbf{J}_\rho$, $\mathbf{M}_\rho$, $\mathbf{H}_\rho$) of the general controller $\hat{C}$ is not unique. Assume that a realisation ($\mathbf{F}_{\rho 0}$, $\mathbf{G}_{\rho 0}$, $\mathbf{J}_{\rho 0}$, $\mathbf{M}_{\rho 0}$, $\mathbf{H}_{\rho 0}$) has been designed through a controller design procedure for $\hat{C}$. All the realisations of $\hat{C}$ form a realisation set:

$$\begin{aligned} \mathcal{S}_\rho \triangleq \{(&\mathbf{F}_\rho, \mathbf{G}_\rho, \mathbf{J}_\rho, \mathbf{M}_\rho, \mathbf{H}_\rho): \\ &\mathbf{F}_\rho = \mathbf{T}_\rho^{-1}\mathbf{F}_{\rho 0}\mathbf{T}_\rho, \mathbf{G}_\rho = \mathbf{T}_\rho^{-1}\mathbf{G}_{\rho 0}, \\ &\mathbf{J}_\rho = \mathbf{J}_{\rho 0}\mathbf{T}_\rho, \mathbf{M}_\rho = \mathbf{M}_{\rho 0}, \mathbf{H}_\rho = \mathbf{T}_\rho^{-1}\mathbf{H}_{\rho 0}\} \end{aligned} \tag{17}$$

where $\mathbf{T}_\rho \in \mathcal{R}^{m\times m}$ is any real-valued, non-singular matrix, called a similarity transformation. Any two realisations in $\mathcal{S}_\rho$ are completely equivalent if they are implemented with infinite precision. Let

$$\mathbf{w}_\rho = \begin{bmatrix} w_{\rho 1} \\ w_{\rho 2} \\ \vdots \\ w_{\rho N} \end{bmatrix} \triangleq \begin{bmatrix} \mathrm{Vec}(\mathbf{F}_\rho) \\ \mathrm{Vec}(\mathbf{G}_\rho) \\ \mathrm{Vec}(\mathbf{J}_\rho) \\ \mathrm{Vec}(\mathbf{M}_\rho) \\ \mathrm{Vec}(\mathbf{H}_\rho) \end{bmatrix}, \quad \mathbf{w}_{\rho 0} \triangleq \begin{bmatrix} \mathrm{Vec}(\mathbf{F}_{\rho 0}) \\ \mathrm{Vec}(\mathbf{G}_{\rho 0}) \\ \mathrm{Vec}(\mathbf{J}_{\rho 0}) \\ \mathrm{Vec}(\mathbf{M}_{\rho 0}) \\ \mathrm{Vec}(\mathbf{H}_{\rho 0}) \end{bmatrix} \tag{18}$$

where $N = (m+p)(m+q) + mp$. We also refer to $\mathbf{w}_\rho$ as a realisation of $\hat{C}$. The stability of the closed-loop system in Fig. 1 depends on the eigenvalues of the transition matrix

$$\begin{aligned} \bar{\mathbf{A}}(\mathbf{w}_\rho) &= \begin{bmatrix} \mathbf{A}_\rho + \mathbf{B}_\rho\mathbf{M}_\rho\mathbf{C}_\rho & \mathbf{B}_\rho\mathbf{J}_\rho \\ \mathbf{G}_\rho\mathbf{C}_\rho + \mathbf{H}_\rho\mathbf{M}_\rho\mathbf{C}_\rho & \mathbf{F}_\rho + \mathbf{H}_\rho\mathbf{J}_\rho \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{T}_\rho^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{A}_\rho + \mathbf{B}_\rho\mathbf{M}_{\rho 0}\mathbf{C}_\rho & \mathbf{B}_\rho\mathbf{J}_{\rho 0} \\ \mathbf{G}_{\rho 0}\mathbf{C}_\rho + \mathbf{H}_{\rho 0}\mathbf{M}_{\rho 0}\mathbf{C}_\rho & \mathbf{F}_{\rho 0} + \mathbf{H}_{\rho 0}\mathbf{J}_{\rho 0} \end{bmatrix} \\ &\quad \times \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{T}_\rho \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{T}_\rho^{-1} \end{bmatrix} \bar{\mathbf{A}}(\mathbf{w}_{\rho 0}) \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{T}_\rho \end{bmatrix} \end{aligned} \tag{19}$$

Let us define the 'stability margin' of $\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))$ as

$$StMa(\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))) \triangleq \begin{cases} 1 - |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))|, & \text{if } \rho = z \\ \dfrac{1}{h} - |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\delta)) + \dfrac{1}{h}|, & \text{if } \rho = \delta \end{cases} \tag{20}$$

It follows, from the fact that the closed-loop system is designed to be stable, that

$$StMa(\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))) = StMa(\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))) > 0, \quad \forall i \in \{1, \ldots, m+n\} \tag{21}$$

which implies that all the different controller realisations $\mathbf{w}_\rho \in \mathcal{S}_\rho$ achieve exactly the same closed-loop poles if they are implemented with infinite precision.

In practice, however, a controller can only be implemented with finite precision. Different realisations will have different FWL characteristics. When $\mathbf{w}_\rho$ is implemented using a fixed-point processor, it is perturbed into $\mathbf{w}_\rho + \Delta\mathbf{w}_\rho$. Assume that the fixed-point processor uses B_f bits for the fractional part of a number. Define

$$\epsilon = 2^{-B_f} \tag{22}$$

Then, each element of $\Delta\mathbf{w}_\rho$ is bounded by $\pm\epsilon/2$, that is

$$\mu(\Delta\mathbf{w}_\rho) \triangleq \max_{i\in\{1,\cdots,N\}} |\Delta w_{\rho i}| \le \frac{\epsilon}{2} \tag{23}$$

With the perturbation $\Delta\mathbf{w}_\rho$, $\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))$ is moved to $\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho + \Delta\mathbf{w}_\rho))$. If an eigenvalue of $\bar{\mathbf{A}}(\mathbf{w}_\rho + \Delta\mathbf{w}_\rho)$ crosses over the stability boundary, the closed-loop system, originally designed to be stable, will become unstable. Intuitively, different controller realisations will have different degrees of robustness to FWL effects. It is highly desired to be able to quantify how robust a controller realisation is in terms of its closed-loop stability under FWL implementation.

3 FWL stability related measure

Roughly speaking, how easily the FWL error $\Delta\mathbf{w}_\rho$ can cause a stable control system to become unstable is determined by how close $\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))$ are to the stability boundary, and how sensitive they are to the controller parameter perturbations. The first factor is determined by the stability margins of the eigenvalues, and the second factor is characterised by the derivatives of the eigenvalues with respect to the controller parameters. In this paper, we consider the following stability related measure [26]:

$$\mu_\rho(\mathbf{w}_\rho) \triangleq \min_{i\in\{1,\cdots,m+n\}} \frac{StMa(\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho)))}{\sum_{j=1}^{N}\left|\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial w_{\rho j}}\right|} \tag{24}$$

Heuristically, the use of $\mu_\rho(\mathbf{w}_\rho)$ as a stability measure of $\mathbf{w}_\rho$ can be justified as follows. When the FWL error $\Delta\mathbf{w}_\rho$ is small, we have

$$\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho+\Delta\mathbf{w}_\rho)) \approx \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho)) + \sum_{j=1}^{N}\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial w_{\rho j}}\Delta w_{\rho j}, \quad \forall i\in\{1,\ldots,m+n\} \tag{25}$$

It then follows that

$$\begin{aligned} &-StMa(\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho+\Delta\mathbf{w}_\rho)))\\ &\quad\le -StMa(\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))) + \sum_{j=1}^{N}\left|\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial w_{\rho j}}\right||\Delta w_{\rho j}|\\ &\quad\le -StMa(\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))) + \mu(\Delta\mathbf{w}_\rho)\sum_{j=1}^{N}\left|\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial w_{\rho j}}\right|, \quad \forall i \end{aligned} \tag{26}$$

If $\mu(\Delta\mathbf{w}_\rho) < \mu_\rho(\mathbf{w}_\rho)$, from eqns. 24 and 26, we have

$$StMa(\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho+\Delta\mathbf{w}_\rho))) > 0 \tag{27}$$

This means that the closed-loop system remains stable under the FWL error $\Delta\mathbf{w}_\rho$. In other words, for a given realisation $\mathbf{w}_\rho$, the closed-loop stability can tolerate those FWL perturbations $\Delta\mathbf{w}_\rho$, whose elements have magnitudes less than $\mu_\rho(\mathbf{w}_\rho)$. The larger $\mu_\rho(\mathbf{w}_\rho)$ is, the larger FWL errors the closed-loop system can tolerate.

The assumption that the controller coefficient perturbations are small is generally valid. For example, with a 10-bit accuracy for B_f, the FWL errors are bounded by 0.5%. The stability related measure $\mu_\rho(\mathbf{w}_\rho)$ is computationally tractable. To compute $\mu_\rho(\mathbf{w}_\rho)$, we need $\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))/\partial w_{\rho j}$, which can be calculated with the following theorem.

Theorem 1: Let $\mathbf{A}=\mathbf{M}_0+\mathbf{M}_1\mathbf{X}\mathbf{M}_2\in\mathcal{R}^{m\times m}$ be diagonalisable, where $\mathbf{X}\in\mathcal{R}^{l\times r}$ and $\mathbf{M}_0$, $\mathbf{M}_1$ and $\mathbf{M}_2$ are independent of $\mathbf{X}$ with proper dimensions. Let $\lambda_i(\mathbf{A})$ denote the ith eigenvalue of $\mathbf{A}$, and let $\mathbf{x}_i(\mathbf{A})$ and $\mathbf{y}_i(\mathbf{A})$ be the right and reciprocal left eigenvectors corresponding to $\lambda_i(\mathbf{A})$, respectively. Then

$$\frac{\partial\lambda_i(\mathbf{A})}{\partial\mathbf{X}} \triangleq \begin{bmatrix} \frac{\partial\lambda_i(\mathbf{A})}{\partial x_{11}} & \cdots & \frac{\partial\lambda_i(\mathbf{A})}{\partial x_{1r}}\\ \vdots & \cdots & \vdots\\ \frac{\partial\lambda_i(\mathbf{A})}{\partial x_{l1}} & \cdots & \frac{\partial\lambda_i(\mathbf{A})}{\partial x_{lr}} \end{bmatrix} = \mathbf{M}_1^T\mathbf{y}_i^*(\mathbf{A})\mathbf{x}_i^T(\mathbf{A})\mathbf{M}_2^T \tag{28}$$

The proof of this theorem can be found in [26].

Remark 1: When $\mathbf{A}$ has no repeated poles, all the eigenvectors corresponding to $\lambda_i(\mathbf{A})$ can be characterised as $\mathbf{x}_i(\mathbf{A})=\eta_i\mathbf{x}_{i0}(\mathbf{A})$, where η_i is a nonzero complex-valued constant and $\mathbf{x}_{i0}(\mathbf{A})$ is a given eigenvector of $\lambda_i(\mathbf{A})$. It is then easy to show that the corresponding reciprocal left eigenvector is $\mathbf{y}_i(\mathbf{A})=1/\eta_i^*\mathbf{y}_{i0}(\mathbf{A})$, with $\mathbf{y}_{i0}(\mathbf{A})$ the reciprocal left eigenvector corresponding to $\mathbf{x}_{i0}(\mathbf{A})$. Therefore, $\mathbf{y}_i^*(\mathbf{A})\mathbf{x}_i^T(\mathbf{A})=\mathbf{y}_{i0}^*(\mathbf{A})\mathbf{x}_{i0}^T(\mathbf{A})$, which means that though each eigenvalue has different eigenvectors, its sensitivity given by eqn. 28 is unique. In the sequel, the closed-loop system is assumed to have no repeated poles.

From eqn. 19, we know that

$$\bar{\mathbf{A}}(\mathbf{w}_\rho) = \begin{bmatrix} \mathbf{A}_\rho+\mathbf{B}_\rho\mathbf{M}_\rho\mathbf{C}_\rho & \mathbf{B}_\rho\mathbf{J}_\rho\\ \mathbf{G}_\rho\mathbf{C}_\rho+\mathbf{H}_\rho\mathbf{M}_\rho\mathbf{C}_\rho & \mathbf{H}_\rho\mathbf{J}_\rho \end{bmatrix} + \begin{bmatrix} 0\\ \mathbf{I} \end{bmatrix}\mathbf{F}_\rho\begin{bmatrix} 0 & \mathbf{I} \end{bmatrix} \tag{29}$$

$$\bar{\mathbf{A}}(\mathbf{w}_\rho) = \begin{bmatrix} \mathbf{A}_\rho+\mathbf{B}_\rho\mathbf{M}_\rho\mathbf{C}_\rho & \mathbf{B}_\rho\mathbf{J}_\rho\\ \mathbf{H}_\rho\mathbf{M}_\rho\mathbf{C}_\rho & \mathbf{F}_\rho+\mathbf{H}_\rho\mathbf{J}_\rho \end{bmatrix} + \begin{bmatrix} 0\\ \mathbf{I} \end{bmatrix}\mathbf{G}_\rho\begin{bmatrix} \mathbf{C}_\rho & 0 \end{bmatrix} \tag{30}$$

$$\bar{\mathbf{A}}(\mathbf{w}_\rho) = \begin{bmatrix} \mathbf{A}_\rho+\mathbf{B}_\rho\mathbf{M}_\rho\mathbf{C}_\rho & 0\\ \mathbf{G}_\rho\mathbf{C}_\rho+\mathbf{H}_\rho\mathbf{M}_\rho\mathbf{C}_\rho & \mathbf{F}_\rho \end{bmatrix} + \begin{bmatrix} \mathbf{B}_\rho\\ \mathbf{H}_\rho \end{bmatrix}\mathbf{J}_\rho\begin{bmatrix} 0 & \mathbf{I} \end{bmatrix} \tag{31}$$

$$\bar{\mathbf{A}}(\mathbf{w}_\rho) = \begin{bmatrix} \mathbf{A}_\rho & \mathbf{B}_\rho\mathbf{J}_\rho\\ \mathbf{G}_\rho\mathbf{C}_\rho & \mathbf{F}_\rho+\mathbf{H}_\rho\mathbf{J}_\rho \end{bmatrix} + \begin{bmatrix} \mathbf{B}_\rho\\ \mathbf{H}_\rho \end{bmatrix}\mathbf{M}_\rho\begin{bmatrix} \mathbf{C}_\rho & 0 \end{bmatrix} \tag{32}$$

$$\bar{\mathbf{A}}(\mathbf{w}_\rho) = \begin{bmatrix} \mathbf{A}_\rho+\mathbf{B}_\rho\mathbf{M}_\rho\mathbf{C}_\rho & \mathbf{B}_\rho\mathbf{J}_\rho\\ \mathbf{G}_\rho\mathbf{C}_\rho & \mathbf{F}_\rho \end{bmatrix} + \begin{bmatrix} 0\\ \mathbf{I} \end{bmatrix}\mathbf{H}_\rho\begin{bmatrix} \mathbf{M}_\rho\mathbf{C}_\rho & \mathbf{J}_\rho \end{bmatrix} \tag{33}$$

Applying theorem 1 gives rise to

$$\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial\mathbf{F}_\rho} = \begin{bmatrix} 0 & \mathbf{I} \end{bmatrix}\mathbf{y}_i^*(\bar{\mathbf{A}}(\mathbf{w}_\rho))\mathbf{x}_i^T(\bar{\mathbf{A}}(\mathbf{w}_\rho))\begin{bmatrix} 0\\ \mathbf{I} \end{bmatrix} \tag{34}$$

$$\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial\mathbf{G}_\rho} = \begin{bmatrix} 0 & \mathbf{I} \end{bmatrix}\mathbf{y}_i^*(\bar{\mathbf{A}}(\mathbf{w}_\rho))\mathbf{x}_i^T(\bar{\mathbf{A}}(\mathbf{w}_\rho))\begin{bmatrix} \mathbf{C}_\rho^T\\ 0 \end{bmatrix} \tag{35}$$

$$\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial\mathbf{J}_\rho} = \begin{bmatrix} \mathbf{B}_\rho^T & \mathbf{H}_\rho^T \end{bmatrix}\mathbf{y}_i^*(\bar{\mathbf{A}}(\mathbf{w}_\rho))\mathbf{x}_i^T(\bar{\mathbf{A}}(\mathbf{w}_\rho))\begin{bmatrix} 0\\ \mathbf{I} \end{bmatrix} \tag{36}$$

$$\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial\mathbf{M}_\rho} = \begin{bmatrix} \mathbf{B}_\rho^T & \mathbf{H}_\rho^T \end{bmatrix}\mathbf{y}_i^*(\bar{\mathbf{A}}(\mathbf{w}_\rho))\mathbf{x}_i^T(\bar{\mathbf{A}}(\mathbf{w}_\rho))\begin{bmatrix} \mathbf{C}_\rho^T\\ 0 \end{bmatrix} \tag{37}$$

$$\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial\mathbf{H}_\rho} = \begin{bmatrix} 0 & \mathbf{I} \end{bmatrix}\mathbf{y}_i^*(\bar{\mathbf{A}}(\mathbf{w}_\rho))\mathbf{x}_i^T(\bar{\mathbf{A}}(\mathbf{w}_\rho))\begin{bmatrix} \mathbf{C}_\rho^T\mathbf{M}_\rho^T\\ \mathbf{J}_\rho^T \end{bmatrix} \tag{38}$$

With these derivatives, $\mu_\rho(\mathbf{w}_\rho)$ can easily be computed using eqn. 24.

It is obvious that the proposed measure (eqn. 24) is a generalisation of the pole sensitivity measure based on an l_1 norm [3]. A brief comparison of this measure with other existing measures is given here. Ultimately, when considering the FWL effects on closed-loop stability, it would be desirable to find the largest open 'sphere' in the controller perturbation space, with size or 'radius' defined by

$$\mu_0(\mathbf{w}_\rho) \triangleq \inf\{\mu(\Delta\mathbf{w}_\rho) : \bar{\mathbf{A}}(\mathbf{w}_\rho+\Delta\mathbf{w}_\rho) \text{ is unstable}\} \tag{39}$$

However, computing the value of $\mu_0(\mathbf{w}_\rho)$ is an unsolved open problem. A practical approach is to consider lower-bound measures of $\mu_0(\mathbf{w}_\rho)$ in some senses, which can be computed easily. Obviously, the closer a computationally tractable measure is to $\mu_0(\mathbf{w}_\rho)$, the better. The pole sensitivity measure based on an l_2 norm [2] is such a lower-bound measure. The measure (eqn. 24) has been shown to be a better lower bound of $\mu_0(\mathbf{w}_\rho)$ than the one based on the l_2 norm [3].

Fialho and Georgiou [6] used the complex stability radius measure to formulate an optimal FWL controller realisation problem that can be represented as a special H_∞ norm minimisation problem [Note 2]. It can also be shown that the measure based on the complex stability radius can be regarded as a lower bound of μ_0 under certain conditions. As these conditions are different from those for the measure (eqn. 24), it is difficult to say which measure is less conservative in estimating the FWL closed-loop stability robustness. It will be case-dependent. We have examples, for some of which the complex stability radius measure produces more accurate results, and for others the measure (eqn. 24) is more accurate. Thus, the complex stability radius measure, like the pole sensitivity approach, is a conservative (lower bound) measure, and the approximation in this case comes from the artificial assumption of complex-valued controller perturbation and the use of a so-called statistical word length formula. The most important advantage of the complex stability radius measure is that the corresponding optimisation problem can be posed as a linear matrix inequality problem, which is easier to solve for than the optimisation problem based on the pole sensitivity approach. The approach based on the complex stability radius measure, however, in its present form, can only be applied to OF controllers with z-operator parameterisation, and it is not known yet how to extend the method to the generic controller structure of Fig. 1.

4 Optimal FWL controller realisation

Since the stability related measure $\mu_\rho(\mathbf{w}_\rho)$ is a function of the controller realisation $\mathbf{w}_\rho$, we can search for an 'optimal' realisation that maximises $\mu_\rho(\mathbf{w}_\rho)$. Such a realisation is optimal in the sense that it has a maximum closed-loop stability robustness to the FWL effects. Given an initial design $(\mathbf{F}_{\rho 0}, \mathbf{G}_{\rho 0}, \mathbf{J}_{\rho 0}, \mathbf{M}_{\rho 0}, \mathbf{H}_{\rho 0})$, any realisation $(\mathbf{F}_\rho, \mathbf{G}_\rho, \mathbf{J}_\rho, \mathbf{M}_\rho, \mathbf{H}_\rho)$ can be characterised with eqn 17. Thus, the optimal controller realisation $\mathbf{w}_{\rho\mathrm{opt}}$ is the solution of the optimisation problem

$$\upsilon_\rho = \max_{\mathbf{w}_\rho \in S_\rho} \mu_\rho(\mathbf{w}_\rho) \tag{40}$$

We now derive the detailed optimisation procedure. $\forall i \in 1, \ldots, m+n$, we partition the eigenvectors of $\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})$, $\mathbf{x}_i(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))$ and $\mathbf{y}_i(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))$, into:

$$\mathbf{x}_i(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) = \begin{bmatrix} \mathbf{x}_{i,1}(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \\ \mathbf{x}_{i,2}(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \end{bmatrix}, \quad \mathbf{y}_i(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) = \begin{bmatrix} \mathbf{y}_{i,1}(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \\ \mathbf{y}_{i,2}(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \end{bmatrix} \tag{41}$$

[Note 2] Fialho and Georgiou's ACC99 paper [6] only contained a two-page summary. A full derivation of the approach is very lengthy and beyond the scope of this paper.

where $\mathbf{x}_{i,1}(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})), \mathbf{y}_{i,1}(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \in C^n$ and $\mathbf{x}_{i,2}(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})), \mathbf{y}_{i,2}(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \in C^m$. It is easy to see from eqn. 19 that, $\forall i \in 1, \ldots, m+n$

$$\mathbf{x}_i(\bar{\mathbf{A}}(\mathbf{w}_\rho)) = \begin{bmatrix} \mathbf{x}_{i,1}(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \\ \mathbf{T}_\rho^{-1}\mathbf{x}_{i,2}(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \end{bmatrix}, \quad \mathbf{y}_i(\bar{\mathbf{A}}(\mathbf{w}_\rho)) = \begin{bmatrix} \mathbf{y}_{i,1}(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \\ \mathbf{T}_\rho^T\mathbf{y}_{i,2}(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \end{bmatrix} \tag{42}$$

where $\mathbf{T}_\rho \in \mathcal{R}^{m\times m}$ and $\det(\mathbf{T}_\rho) \neq 0$. Applying eqn. 42 to eqns. 34–38 results in

$$\frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial \mathbf{F}_\rho} = \mathbf{T}_\rho^T \mathbf{y}_{i,2}^*(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))\mathbf{x}_{i,2}^T(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))\ \mathbf{T}_\rho^{-T} \tag{43}$$

$$\frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial \mathbf{G}_\rho} = \mathbf{T}_\rho^T \mathbf{y}_{i,2}^*(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))\ \mathbf{x}_{i,1}^T(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))\ \mathbf{C}_\rho^T \tag{44}$$

$$\begin{aligned} \frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial \mathbf{J}_\rho} &= (\mathbf{B}_\rho^T \mathbf{y}_{i,1}^*(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \\ &\quad + \mathbf{H}_\rho^T \mathbf{T}_\rho^T \mathbf{y}_{i,2}^*(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})))\mathbf{x}_{i,2}^T(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))\ \mathbf{T}_\rho^{-T} \\ &= (\mathbf{B}_\rho^T \mathbf{y}_{i,1}^*(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \\ &\quad + \mathbf{H}_{\rho 0}^T \mathbf{y}_{i,2}^*(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})))\mathbf{x}_{i,2}^T(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))\ \mathbf{T}_\rho^{-T} \end{aligned} \tag{45}$$

$$\begin{aligned} \frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial \mathbf{M}_\rho} &= (\mathbf{B}_\rho^T \mathbf{y}_{i,1}^*(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \\ &\quad + \mathbf{H}_\rho^T \mathbf{T}_\rho^T \mathbf{y}_{i,2}^*(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})))\mathbf{x}_{i,1}^T(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))\ \mathbf{C}_\rho^T \\ &= (\mathbf{B}_\rho^T \mathbf{y}_{i,1}^*(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})) \\ &\quad + \mathbf{H}_{\rho 0}^T \mathbf{y}_{i,2}^*(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})))\mathbf{x}_{i,1}^T(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))\ \mathbf{C}_\rho^T \end{aligned} \tag{46}$$

$$\begin{aligned} \frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial \mathbf{H}_\rho} &= \mathbf{T}_\rho^T \mathbf{y}_{i,2}^*(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))(\mathbf{x}_{i,1}^T(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))\mathbf{C}_\rho^T \mathbf{M}_\rho^T \\ &\quad + \mathbf{x}_{i,2}^T(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))\mathbf{T}_\rho^{-T}\mathbf{J}_\rho^T) \\ &= \mathbf{T}_\rho^T \mathbf{y}_{i,2}^*(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))(\mathbf{x}_{i,1}^T(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))\mathbf{C}_\rho^T \mathbf{M}_{\rho 0}^T \\ &\quad + \mathbf{x}_{i,2}^T(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0}))\ \mathbf{J}_{\rho 0}^T) \end{aligned} \tag{47}$$

Define the following function of the similarity matrix $\mathbf{T}_\rho$:

$$f_\rho(\mathbf{T}_\rho) \triangleq \mu_\rho(\mathbf{w}_\rho) = \min_{i\in\{1,\cdots,m+n\}} \frac{StMa(\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_{\rho 0})))}{\alpha_{\rho i}(\mathbf{w}_\rho) + \beta_{\rho i}(\mathbf{w}_\rho)} \tag{48}$$

where

$$\alpha_{\rho i}(\mathbf{w}_\rho) = \left\|\frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial \mathbf{F}_\rho}\right\|_S + \left\|\frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial \mathbf{G}_\rho}\right\|_S + \left\|\frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial \mathbf{H}_\rho}\right\|_S$$

and

$$\beta_{\rho i}(\mathbf{w}_\rho) = \left\|\frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial \mathbf{J}_\rho}\right\|_S + \left\|\frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\rho))}{\partial \mathbf{M}_\rho}\right\|_S$$

Then the problem (eqn. 40) of finding an optimal controller realisation $\mathbf{w}_{\rho\mathrm{opt}}$ is equivalent to obtaining an optimal similarity matrix that is the solution of the following nonlinear optimisation problem:

$$\mathbf{T}_{\rho\mathrm{opt}} = \arg \max_{\substack{\mathbf{T}_\rho \in \mathcal{R}^{m\times m} \\ \det(\mathbf{T}_\rho)\neq 0}} f_\rho(\mathbf{T}_\rho) \tag{49}$$

To find a $\mathbf{T}_{\rho\text{opt}}$, we will adopt an iterative optimisation procedure to generate a sequence $\{\mathbf{T}_{\rho 0}, \mathbf{T}_{\rho 1}, \ldots\}$, which converges to $\mathbf{T}_{\rho\text{opt}}$. Define $\Omega \triangleq \{\mathbf{T}_\rho \in \mathcal{R}^{m\times m}: \det(\mathbf{T}_\rho) = 0\}$. As Ω is only a manifold in $\mathcal{R}^{m\times m}$, starting from a $\mathbf{T}_{\rho 0} \notin \Omega$, it is rare for an iterative sequence $\{\mathbf{T}_{\rho i}\}$ to move into Ω. Thus, in the iterative procedure, the constraint $\det(\mathbf{T}_\rho) \neq 0$ can practically be ignored, leading to an 'unconstrained' optimisation problem:

$$\max_{\mathbf{T}_\rho \in \mathcal{R}^{m\times m}} f_\rho(\mathbf{T}_\rho) \tag{50}$$

The possible pitfall of violating the constraint can readily be avoided by monitoring the singular values of $\mathbf{T}_\rho$. If a singular value of $\mathbf{T}_\rho$ is too small, a small perturbation $\eta\mathbf{I}$ is added to $\mathbf{T}_\rho$ so that $\mathbf{T}_\rho + \eta\mathbf{I} \notin \Omega$. This small perturbation, which is rarely needed, will not affect the convergence of the iterative procedure. Because $f_\rho(\mathbf{T}_\rho)$ is non-smooth and non-convex, optimisation must be based on a direct search without the aid of cost function derivatives. The conventional optimisation methods for this kind of problem, such as Rosenbrock and Simplex algorithms [7, 8], generally can only find a local minimum. We will adopt an efficient global optimisation strategy based on the adaptive simulated annealing (ASA) algorithm [12–14] to search for a true global optimum $\mathbf{T}_{\rho\text{opt}}$. With $\mathbf{T}_{\rho\text{opt}}$, we can readily obtain the optimal controller realisation $\mathbf{w}_{\rho\text{opt}}$. The detailed implementation of the ASA algorithm is given in [14].

5 Comparison between z and δ realisations

The z-operator controller realisation $\mathbf{w}_z$ is completely equivalent to the δ-operator realisation $\mathbf{w}_\delta$ under infinite-precision implementation. We analyse the underlying relationship between these two parameterisations of the controller structure and investigate their FWL implementation characteristics. We will assume that h in the δ operator has an exact FWL representation, e.g., $h = 2^2$, $h = 2^{-6}$. Thus, the source of FWL errors comes solely from the FWL implementation of $\mathbf{w}_\delta$. Define a map g_h from $\mathcal{S}_z$ to $\mathcal{S}_\delta$:

$$\mathbf{w}_\delta = g_h(\mathbf{w}_z) \Leftrightarrow \begin{cases} \mathbf{F}_\delta = \dfrac{\mathbf{F}_z - \mathbf{I}}{h} \\ \mathbf{G}_\delta = \dfrac{\mathbf{G}_z}{h} \\ \mathbf{J}_\delta = \mathbf{J}_z \\ \mathbf{M}_\delta = \mathbf{M}_z \\ \mathbf{H}_\delta = \dfrac{\mathbf{H}_z}{h} \end{cases} \tag{51}$$

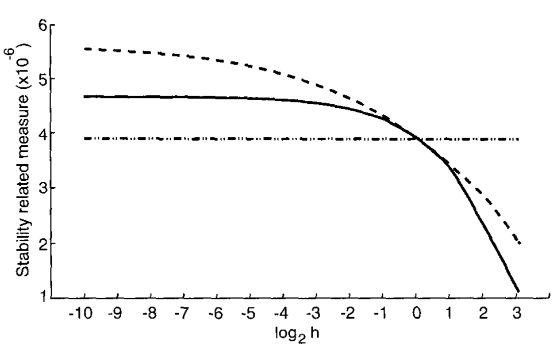

Fig. 2 *Comparison of values of stability related measure for optimal δ realisation $\mathbf{w}_{\delta opt}$, δ realisation $\mathbf{w}_\delta = g_h(\mathbf{w}_{zopt})$ and optimal z realisation $\mathbf{w}_{zopt}$*
--- $v_\delta(h)$
—— $f(h)$
-·-·- v_z

We can see that g_h is a one-to-one map.

Lemma 4: $\mu_\delta(g_h(\mathbf{w}_z)) \geq \mu_z(\mathbf{w}_z)$ when $h < 1$; $\mu_\delta(g_h(\mathbf{w}_z)) = \mu_z(\mathbf{w}_z)$ when $h = 1$; $\mu_\delta(g_h(\mathbf{w}_z)) \leq \mu_z(\mathbf{w}_z)$ when $h > 1$.

Proof. For $\mathbf{w}_\delta = g_h(\mathbf{w}_z)$, it follows from lemma 2 and remark 1 that

$$\mathbf{y}_i^*(\bar{\mathbf{A}}(\mathbf{w}_\delta))\mathbf{x}_i^T(\bar{\mathbf{A}}(\mathbf{w}_\delta)) = \mathbf{y}_i^*(\bar{\mathbf{A}}(\mathbf{w}_z))\mathbf{x}_i^T(\bar{\mathbf{A}}(\mathbf{w}_z)) \tag{52}$$

Noting eqn. 51 and, for the plant, $\mathbf{A}_\delta = (\mathbf{A}_z - \mathbf{I})/h$, $\mathbf{B}_\delta = \mathbf{B}_z/h$ and $\mathbf{C}_\delta = \mathbf{C}_z$, it then follows from eqns 34–38 that

$$\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\delta))}{\partial\mathbf{F}_\delta} = \frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))}{\partial\mathbf{F}_z} \tag{53}$$

$$\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\delta))}{\partial\mathbf{G}_\delta} = \frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))}{\partial\mathbf{G}_z} \tag{54}$$

$$\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\delta))}{\partial\mathbf{J}_\delta} = \frac{1}{h}\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))}{\partial\mathbf{J}_z} \tag{55}$$

$$\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\delta))}{\partial\mathbf{M}_\delta} = \frac{1}{h}\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))}{\partial\mathbf{M}_z} \tag{56}$$

$$\frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\delta))}{\partial\mathbf{H}_\delta} = \frac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))}{\partial\mathbf{H}_z} \tag{57}$$

Hence, $\forall i \in \{1, \ldots, m+n\}$:

$$\frac{\dfrac{1}{h} - \left|\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\delta)) + \dfrac{1}{h}\right|}{\sum_{j=1}^N \left|\dfrac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\delta))}{\partial w_{\delta j}}\right|} = \frac{\dfrac{1}{h} - \left|\dfrac{\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))}{h}\right|}{\alpha_{\delta i}(\mathbf{w}_\delta) + \beta_{\delta i}(\mathbf{w}_\delta)} = \frac{1 - |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))|}{h\alpha_{zi}(\mathbf{w}_z) + \beta_{zi}(\mathbf{w}_z)} \tag{58}$$

On the other hand, $\forall i \in \{1, \ldots, m+n\}$:

$$\frac{1 - |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))|}{\sum_{j=1}^N \left|\dfrac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))}{\partial w_{zj}}\right|} = \frac{1 - |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))|}{\alpha_{zi}(\mathbf{w}_z) + \beta_{zi}(\mathbf{w}_z)} \tag{59}$$

When $h < 1$, comparing eqn. (58) with eqn. 59 leads to

$$\frac{\dfrac{1}{h} - \left|\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\delta)) + \dfrac{1}{h}\right|}{\sum_{j=1}^N \left|\dfrac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_\delta))}{\partial w_{\delta j}}\right|} \geq \frac{1 - |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))|}{\sum_{j=1}^N \left|\dfrac{\partial\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_z))}{\partial w_{zj}}\right|}, \quad \forall i \in \{1, \ldots, m+n\} \tag{60}$$

which means that $\mu_\delta(g_h(\mathbf{w}_z)) \geq \mu_z(\mathbf{w}_z)$. The results for $h = 1$ and $h > 1$ can similarly be proved.

For $\upsilon_z = \mu_z(\mathbf{w}_{z\text{opt}})$ and $\upsilon_\delta = \mu_\delta(\mathbf{w}_{\delta\text{opt}})$, based on lemma 4, we have the following.

Corollary 1: $\upsilon_\delta \geq \upsilon_z$ when $h < 1$; $\upsilon_\delta = \upsilon_z$ when $h = 1$; $\upsilon_\delta \leq \upsilon_z$ when $h > 1$.

Corollary 1 shows that if h is chosen to be smaller than 1, the optimal δ realisation has better FWL stability characteristics than the optimal z realisation; if h is chosen to be larger than 1, the optimal δ realisation has worse FWL stability characteristics than the optimal z realisation; if h is chosen to be equal to 1, both optimal realisations have the same FWL stability robustness to the FWL effects. We notice that δ realisations are dependent of h while z realisations are independent of h. Thus, υ_δ is a function of h, which will be denoted as $v_\delta(h)$, while v_z is not. Let us introduce the function

$$f(h) = \min_{i \in \{1,\cdots,m+n\}} \frac{\kappa_i}{h\alpha_i + \beta_i} \quad (61)$$

where

$$\kappa_i = 1 - |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_{zopt}))| \quad (62)$$

$$\alpha_i = \left\| \frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_{zopt}))}{\partial \mathbf{F}_z} \right\|_S + \left\| \frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_{zopt}))}{\partial \mathbf{G}_z} \right\|_S + \left\| \frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_{zopt}))}{\partial \mathbf{H}_z} \right\|_S \quad (63)$$

and

$$\beta_i = \left\| \frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_{zopt}))}{\partial \mathbf{J}_z} \right\|_S + \left\| \frac{\partial \lambda_i(\bar{\mathbf{A}}(\mathbf{w}_{zopt}))}{\partial \mathbf{M}_z} \right\|_S \quad (64)$$

Theorem 2: $v_z = f(1)$ and $v_\delta(h) \geq f(h)$.

Proof: $v_z = f(1)$ can be directly obtained from the definitions of $\mu_z(\mathbf{w}_{zopt})$ and $f(h)$. From the proof of lemma 4, it can easily be seen that $\mu_\delta(g_h(\mathbf{w}_{zopt})) = f(h)$. Noting $v_\delta(h) = \max_{\mathbf{w}_\delta \in S_\delta} \mu_\delta(\mathbf{w}_\delta) \geq \mu_\delta(g_h(\mathbf{w}_{zopt}))$, we conclude that $v_\delta(h) \geq f(h)$.

Notice that $f(h)$ is defined in $(0, \infty)$ and $f(h)$ decreases as h increases. According to theorem 2, for $h \in (0, 1)$, the optimal δ realisation has better FWL closed-loop stability performance than the optimal z realisation and, furthermore, the smaller h is, the larger $v_\delta(h)$ is than v_z. It is well known that, when $h \to 0$, the δ-operator representation approaches the continuous-time representation. It is therefore expected that $f(h)$, and hence $v_\delta(h)$, will approach certain limit values as $h \to 0$.

6 Design example

We present a numerical example to illustrate the proposed optimisation approach and verify the theoretical results given in the previous Section. The plant model used is a modification of the plant studied in [2], which was a single-input, single-output system. We have added one more output, which is the first state in the original plant model. The state-space model of this modified plant, represented in the z-operator, is given by:

$$\mathbf{A}_z = \begin{bmatrix} 3.2439e-01 & -4.5451e+00 & -4.0535e+00 & -2.7003e-03 & 0 \\ 1.4518e-01 & 4.9477e-01 & -4.6945e-01 & -3.1274e-04 & 0 \\ 1.6814e-02 & 1.6491e-01 & 9.6681e-01 & -2.2114e-05 & 0 \\ 1.1889e-03 & 1.8209e-02 & 1.9829e-01 & 1.0000e+00 & 0 \\ 6.1301e-05 & 1.2609e-03 & 1.9930e-02 & 2.0000e-01 & 1 \end{bmatrix}$$

$$\mathbf{B}_z = \begin{bmatrix} 1.4518e-01 \\ 1.6814e-02 \\ 1.1889e-03 \\ 6.1301e-05 \\ 2.4979e-06 \end{bmatrix}$$

$$\mathbf{C}_z = \begin{bmatrix} 0 & 0 & 1.6188e+00 & -1.5750e-01 & -4.3943e+01 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The closed-loop poles as given in [2] were used in the design, and the designed controller obtained using a standard design procedure [18] had a state-space form:

$$\mathbf{F}_{z0} = \begin{bmatrix} 0 & 1 \\ -9.3303e-01 & 1.9319e+00 \end{bmatrix},$$

$$\mathbf{G}_{z0} = \begin{bmatrix} 4.1814e-02 & 2.7132e+02 \\ 3.9090e-02 & 1.0167e+03 \end{bmatrix}$$

$$\mathbf{J}_{z0} = [3.0000e-04 \quad 5.0000e-04],$$

$$\mathbf{M}_{z0} = [0 \quad 6.1250e-01], \quad \mathbf{H}_{z0} = \begin{bmatrix} 7.8047e+01 \\ 7.3849e+01 \end{bmatrix}$$

With this initial realisation $\mathbf{w}_{z0}$, the corresponding transition matrix $\bar{\mathbf{A}}(\mathbf{w}_{z0})$ was formed using eqn. 19, from which the poles and the eigenvectors of the ideal closed-loop system were computed. The value of the stability related measure for $\mathbf{w}_{z0}$ is $\mu_z(\mathbf{w}_{z0}) = 4.0509e-07$.

Using the ASA algorithm to solve for the resulting optimisation problem (eqn. 49) gave rise to the following optimal similarity transformation matrix:

$$\mathbf{T}_{zopt} = \begin{bmatrix} -1.7791e+01 & 3.5665e+00 \\ -1.6696e+01 & 3.5384e+00 \end{bmatrix}$$

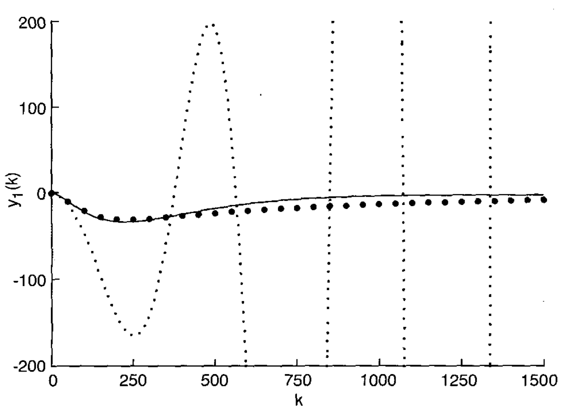

Fig. 3 *Comparison of unit impulse response for ideal infinite-precision controller implementation* $\mathbf{w}_{ideal}$ *with those for two 10-bit implemented controller realisations* $\mathbf{w}_{z0}$ *and* $\mathbf{w}_{zopt}$

—— $\mathbf{w}_{ideal}$
····· $\mathbf{w}_{z0}$
••• $\mathbf{w}_{zopt}$

The optimal z realisation corresponding to $\mathbf{T}_{zopt}$ was

$$\mathbf{F}_{zopt} = \begin{bmatrix} 9.5253e-01 & -2.5578e-03 \\ 7.0338e-02 & 9.7934e-01 \end{bmatrix},$$

$$\mathbf{G}_{zopt} = \begin{bmatrix} -2.5073e-03 & 7.8274e+02 \\ -7.8313e-04 & 3.9806e+03 \end{bmatrix}$$

$$\mathbf{J}_{zopt} = [-1.3685e-02 \quad 2.8392e-03],$$

$$\mathbf{M}_{zopt} = [0 \quad 6.1250e-01], \quad \mathbf{H}_{zopt} = \begin{bmatrix} -3.7504e+00 \\ 3.1750e+00 \end{bmatrix}$$

The optimal stability related measure was $v_z = \mu_z(\mathbf{w}_{zopt}) = 3.8927e-06$. This represents an improvement by approximately a factor of ten over the initial controller realisation.

Similarly, we constructed and solved for the optimal δ realisation problem for $h = 2^3 \sim 2^{-10}$. Fig. 2 compares $v_\delta(h)$, the stability related measure for the optimal δ realisation, with $f(h)$ and v_z. It can be seen that the results of Fig. 2 agree with the theoretical analysis of corollary 1 and theorem 2. As expected, for $h < 1$, the optimal δ realisation has a larger FWL closed-loop stability measure than the optimal z realisation.

We also computed the unit impulse response of the closed-loop control system when the controllers were the infinite-precision implemented $\mathbf{w}_{z0}$ and various FWL implemented realisations with 10-bit accuracy for B_f, respectively. Note that any realisation $\mathbf{w}_\rho \in \mathcal{S}_\rho$, implemented in infinite precision, will achieve the exact performance of the infinite-precision implemented $\mathbf{w}_{z0}$, which is the *designed* controller performance. For this reason, the infinite-precision implemented $\mathbf{w}_{z0}$ is referred to as the *ideal* controller realisation $\mathbf{w}_{\text{ideal}}$. Figs. 3–6 compare the unit impulse response of the first plant output $y_1(k)$ for the ideal controller $\mathbf{w}_{\text{ideal}}$ with those of various 10-bit implemented realisations. It can be seen that the closed-loop became unstable with a 10-bit implemented controller realisation $\mathbf{w}_{z0}$. The results also clearly show the benefits of the proposed optimisation process, as the closed-loop system remained stable with the 10-bit implemented $\mathbf{w}_{zopt}$ Furthermore, the 10-bit implemented $\mathbf{w}_{\delta\text{opt}}$ with $h = 2^{-1}$ was able to approximate closely the designed performance

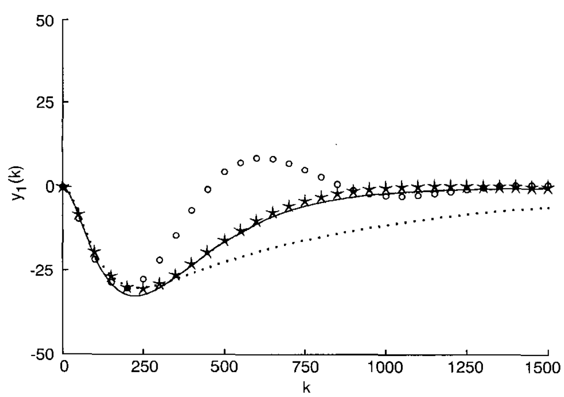

Fig. 4 *Comparison of unit impulse response for ideal infinite-precision controller implementation $\mathbf{w}_{ideal}$ with those for three 10-bit implemented controller realisations $\mathbf{w}_{zopt}$, $\mathbf{w}_{\delta opt}$ ($h = 2^2$), and $\mathbf{w}_{\delta opt}$ ($h = 2^{-1}$)*

—— $\mathbf{w}_{\text{ideal}}$
····· $\mathbf{w}_{\text{zopt}}$
○○○ $\mathbf{w}_{\delta\text{opt}}(h = 2^2)$
∗∗∗ $\mathbf{w}_{\delta\text{opt}}(h = 2^{-1})$

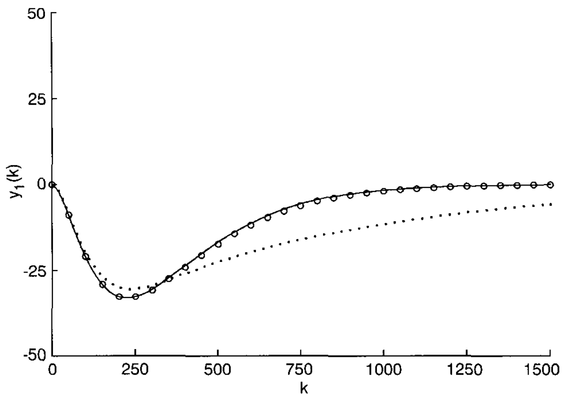

Fig. 5 *Comparison of unit impulse response for ideal infinite-precision controller implementation $\mathbf{w}_{ideal}$ with those for two 10-bit implemented controller realisations $\mathbf{w}_{zopt}$ and $\mathbf{w}_{\delta opt}$ ($h = 2^{-7}$)*

—— $\mathbf{w}_{\text{ideal}}$
····· $\mathbf{w}_{\text{zopt}}$
○○○ $\mathbf{w}_{\delta\text{opt}}$ $(h = 2^{-7})$

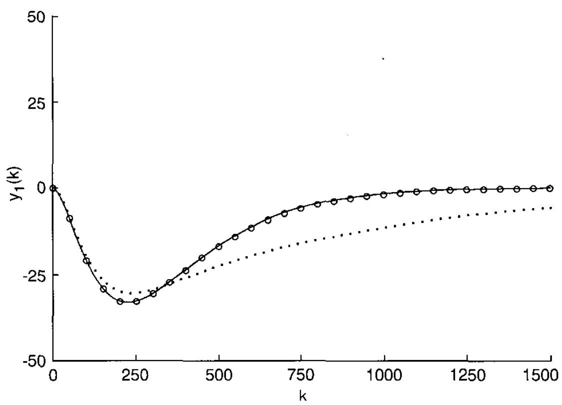

Fig. 6 *Comparison of unit impulse response for ideal infinite-precision controller implementation $\mathbf{w}_{ideal}$ with those for two 10-bit implemented controller realisations $\mathbf{w}_{zopt}$ and $\mathbf{w}_{\delta opt}$ ($h = 2^{-10}$)*

—— $\mathbf{w}_{\text{ideal}}$
····· $\mathbf{w}_{\text{zopt}}$
○○○ $\mathbf{w}_{\delta\text{opt}}(h = 2^{-10})$

of the ideal infinite-precision controller. With h reduced to 2^{-7}, the 10-bit implemented $\mathbf{w}_{\delta\text{opt}}$ achieved the designed controller performance.

7 Conclusions

We have studied the finite-precision implementation issues for digital controllers. A unified approach has been adopted to derive a tractable FWL closed-loop stability related measure for both the z and δ-operator parameterisations of the general controller structure. An efficient optimisation procedure has been developed for obtaining the optimal controller realisation that maximises the proposed measure. The underlying relationship connecting the z and δ realisations has been investigated. Because the FWL stability measure for δ controller realisation is a function of the operator constant h, we can always obtain an optimal δ controller realisation that has a better closed-loop stability margin than the optimal z realisation in the FWL implementation. The theoretical results have been verified and the optimisation procedure demonstrated using a numerical design example.

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