Proceedings of the 14th International Conference
on Automation & Computing, Brunel University,
West London, UK, 6 September 2008

Stability of Networked Control Systems with Random Buffer Capacity

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***Abstract*— Stability of the discrete-time networked control systems is analysed, where the controller is updated with the buffered sensor information at stochastic intervals and the amount of the buffered data for transmission under the buffer capacity constraint is time-varying. The adopted controller switches between open-loop and closed-loop modes to estimate the plant behaviour. The sufficient condition for the Lyapunov stability with the generic arbitrary transmission is derived, and the sufficient conditions for the mean square stability with the Markovian transmission are also established. An example is given to demonstrate the effectiveness of our method.**

***Index Terms*— Networked control systems, buffer capacity constraint, Lyapunov stability, mean square stability**

I. Introduction

Networked control systems (NCSs) have received much attention recently, see [1], [2], [3] and the references therein. A NCS is a control system in which a control loop is closed via a shared communication network. The use of a shared network in the feedback path offers several advantages, including low installation cost, reducing system wiring, simple system diagnosis and easy maintenance. However, the NCS also has some inherent shortcomings, such as bandwidth constraints, packet delays and packet dropout, which degrade system performance or even cause closed-loop instability. Guaranteeing closed-loop stability is an upmost requirement for any NCS. Stability analysis of NCSs is investigated in [4], [5], [6], [7], [8], and stabilising controllers are designed in [9], [10], [11], [12], [13]. Stochastic approaches are generally adopted to deal with network packet dropout, and these approaches attempt to establish the stability in terms of mean square stability [14], [15]. In the literature, the characteristic of the network is usually modeled as a Markov process and the system is considered as a special case of discrete-time Markovian jump linear system (MJLS) [9], [11], [13]. Many different networks have been promoted for use in control systems [16], including specialised networks such as CAN, ControlNet, FIP and Profibus. However, there has been a growing trend to move towards the general-purpose networks such as Ethernet and wireless LANs.

A festure of modern communication protocols is that data is sent in large packets. For example, in Ethernet the frame format allows for a packet of 46 to 1500 bytes [17]. For IEEE 802.11, the packet size in each frame is up to 2312 bytes [18]. This opens up the possibility to design control schemes in which a large amount of data, consisting the current and past values, rather than just the most recent value, are sent through the network at the transmission instant. To achieve this goal, a buffer is needed to store information for transmission. The works of [19], [20], [21] adopt predictive control approach with this transmission strategy implemented between the cotroller and the actuator. Specifically, in [19] the authors adopt the predictive control to analyse the stability of NCSs with both fixed and random network transmission delays, while in [20] the authors consider a packetised predictive networked control scheme in which an optimising sequence of control inputs is sent over a communication network affected by packet dropouts. In [21] the authors study robust H_∞ controllers for NCSs with both networked-induced delay and packet dropout by predictive method. In all these three works, the buffer length is assumed to be sufficiently large, i.e. no buffer capacity limit, so that the length of the sequence of data for transmission can be kept constant. The works of [10], [22] implement this transmission strategy between the sensor and the controller. In [10] the stability of the NCS is obtained under the condition that the length of the data sequence for transmission is no less than a lower bound together with the assumpton that the system is nominal, i.e. no plant model uncertainty. In [22] authors demonstrate that sending a linear combination of two measurements, past and present, which minimises the state estimation error covariance, is better than sending the most recent observation.

In practice, the buffer length is finite. Due to the network induced delay and packet dropout, the amount of data stored in the buffer at transmission instant is inherently random, varying between 1 to the buffer capacity. To the best of our knowledge, no work to date addresses this practical problem of random buffer capacity constraint. The novelty of this paper is that we consider the stability of the NCS with this random buffer capacity constraint. In our system, the network separates the sensor/buffer and the controller, the buffer capacity is limited and, therefore, the length of the data sequence for transmission is time-varying. We also consider plant model uncertainty. Stochastic update intervals between consecutively successfully transmissions are categorised into two types. One type is the generic transmission which takes values from a finite set arbitrarily, and the common Lyapunov function is adopted here to establish the stability condition for this case. The other type is the Markovian transmission, in which the update interval is driven by a discrete-time first-order Markov chain with the known transition probability matrix, and the mean square stability conditions are derived for this case. In our approach, the control law is switched, which can better adapt to the characteristics of the network-induced delay and packet dropout. Specifically, a smart controller, which is collocated with the actuator, is updated with the random-length data packets received at stochastic time intervals as well as the historic information of the controller, and switches the structure between open loop and closed loop to estimate the plant behaviour.

Throughout this contribution we adopt the following notational conventions. $\mathbb{R}$ stands for real numbers and $\mathbb{N}$ for nonnegative integers. For a given matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and vector $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\| \triangleq \sqrt{\mathbf{x}^T\mathbf{x}}$ denotes the Euclidean norm of $\mathbf{x}$, $\|\mathbf{A}\|$ the corresponding induced matrix norm, and $\bar{\sigma}(\mathbf{A})$ the largest singular value of $\mathbf{A}$. $\mathbf{W} > \mathbf{0}$ indicates that $\mathbf{W}$ is a symmetric positive-definite matrix. $\mathbf{I}$ and $\mathbf{0}$ represent the identity and zero matrices of appropriate dimensions, respectively. Finally $E\{\mathbf{y}(t)\}$ defines the expectation of $\mathbf{y}(t)$.

II. Problem formulation

The NCS $\hat{P}_K$ of Fig. 1 contains a discrete-time plant $\hat{P}$ and a discrete-time controller $\hat{K}$ with the control loop closed via a shared communication network. The plant $\hat{P}$ is described by the state-space equation

$$\begin{cases} \mathbf{x}(t+1) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \end{cases} \quad t \in \mathbb{N} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$ and $\mathbf{y}(t) \in \mathbb{R}^p$ are the state, input and output vectors of the system, respectively, while $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{p\times n}$ are the true matrices of the plant state-space equation. We only have the matrices of the plant model $\hat{\mathbf{A}} \in \mathbb{R}^{n\times n}$, $\hat{\mathbf{B}} \in \mathbb{R}^{n\times m}$ and $\hat{\mathbf{C}} \in \mathbb{R}^{p\times n}$, which can be different from $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. The network is between the sensor and the controller, while the controller is collocated with the actuator. $\hat{P}$ and $\hat{K}$ are time-driven and synchronised. Thus, the sensor and the controller have the same sampling period T. It is further assumed that the actual packet transmission delay through the network is smaller than T and, therefore, it is negligible. Hence we mainly consider the packet dropout and the network delay induced by the waiting time at the buffer node. The plant outputs, $\{\mathbf{y}(t)\}$, are stored in the buffer with a maximum capacity of $q_{\max}$ until they are transmitted. We will use the notation $\hat{\mathbf{y}}(t)$ to denote the sequence of plant outputs that are stored in the buffer.

At each sampling instant $t \in \mathbb{N}$, the buffer seeks for the authorisation to transmit $\hat{\mathbf{y}}(t)$ to $\hat{K}$ through the network. If the authorisation arrives, transmission takes place and after the transmission, $\hat{\mathbf{y}}(t)$ is discarded by the buffer. There are two alternative outcomes of a transmission: one is that transmission succeeds and $\hat{K}$ receives $\hat{\mathbf{y}}(t)$ at t; the other is that the transmission fails due to the packet dropout and $\hat{K}$ misses $\hat{\mathbf{y}}(t)$. Also the effect of discarding data by the buffer due to reaching the capacity $q_{\max}$ has the same effect of network packet dropout. Those instants at which transmissions succeed are denoted as t_k in ascending order,

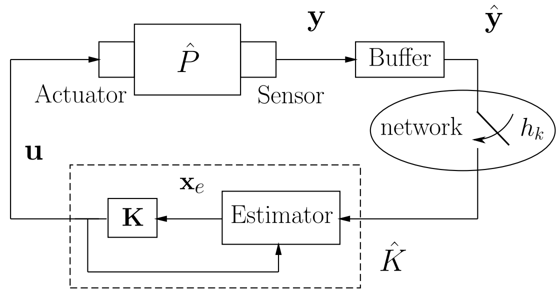

Fig. 1. Networked control system $\hat{P}_K$.

and $t_0 = -1$ is assumed without loss of generality. As $\hat{K}$ receives the new information $\hat{\mathbf{y}}(t_k)$ at t_k, t_k is referred to as the update instant. Note that the length of $\hat{\mathbf{y}}(t_k)$ is stochastic. Define the update interval

$$h_k \triangleq t_k - t_{k-1}, \tag{2}$$

which can take values from a finite set

$$h_k \in \mathcal{N} \triangleq \{1, \ldots, N\} \tag{3}$$

with the maximal update interval N. The value of N can be viewed as a "network quality" measure. When the network is very busy, experiencing long delay and a large number of packet dropouts, N will be very large. By contrast, a small N shows that the network is offering "good-quality" service. The update interval h_k is an integer-valued random process. Two cases of successful transmissions are considered. The transmission is said to be arbitrary if h_k takes values in $\mathcal{N}$ arbitrarily. The arbitrary transmission is the general case where no particular probability distribution is imposed. If a Markovian probability distribution is assumed, the transmission is said to be Markovian. Thus, for the Markovian transmission case, h_k is driven by a discrete-time Markov chain and takes values in $\mathcal{N}$ with the known transition probability matrix $\Gamma \triangleq (p_{ij}) \in \mathbb{R}^{N\times N}$, where $p_{ij} = \text{Prob}(h_{k+1} = j | h_k = i)$, $p_{ij} \geq 0$ and $\sum_{j=1}^{N} p_{ij} = 1$ for each $i \in \mathcal{N}$. Denote h_0 as the initial condition of the Markov chain. The initial state probabilities $\text{Prob}(h_0 = i)$ for all $i \in \mathcal{N}$ are assumed to be equal.

Define q_k as the length of the buffered data which are transmitted at t_k. Then $\hat{\mathbf{y}}(t_k)$ includes the sequence of plant outputs as

$$\hat{\mathbf{y}}(t_k) = \big\{\mathbf{y}(t_k), \mathbf{y}(t_k - 1), \cdots, \mathbf{y}(t_k - q_k + 1)\big\}. \tag{4}$$

That is, q_k is the length of $\hat{\mathbf{y}}(t_k)$ that the controller receives at t_k, and q_k can take values from the finite set

$$q_k \in \mathcal{Q}_k \triangleq \{1, \ldots, s_k\} \tag{5}$$

where

$$s_k \triangleq \min\{q_{\max},\ h_k\}. \tag{6}$$

Because the network induced waiting time and packet dropout are random, we assume that q_k is independent and uniformly distributed. Thus, the state probabilities of q_k are

$$\text{Prob}(q_k = r) = \frac{1}{s_k},\ r \in \mathcal{Q}_k. \tag{7}$$

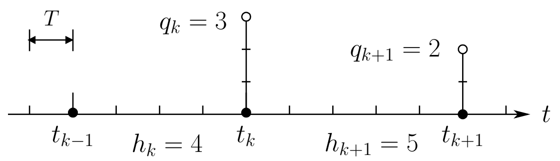

Fig. 2. Time diagram of NCS with buffer status at t_k and t_{k+1}, assuming $q_{\max} = 3$, and packet dropout occurs at $t_{k-1}+1$ and $t_k + 3$.

Note that if packet dropout occurs during the interval h_k, $q_k < s_k$; otherwise $q_k = s_k$. An illustration of this random buffer length is illustrated in Fig. 2.

The controller $\hat{K}$ of the NCS consists of a state feedback gain $\mathbf{K} \in \mathbb{R}^{m\times n}$ and a switched estimator with an estimator gain matrix $\mathbf{L} \in \mathbb{R}^{n\times p}$. The controller output is given by

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}_e(t) \tag{8}$$

where $\mathbf{x}_e(t) \in \mathbb{R}^n$ is the estimator state. From the previous discussion, it can be seen that for $t \neq t_k$ the feedback loop is effectively broken. Thus, the system $\hat{P}_K$ is in the mode of open loop for $t \neq t_k$, while for $t = t_k$ $\hat{P}_K$ is in the mode of closed loop. The switched estimator is designed to adapt to these two different characteristics of the NCS. Our objective is to derive the stability criterion for the NCS $\hat{P}_K$ with given $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, $\mathbf{K}$, $\mathbf{L}$, $\mathcal{N}$, $q_{\max}$ and Γ.

III. Networked Control System Response

For $t \neq t_k$, the estimator takes the form

$$\mathbf{x}_e(t+1) = \hat{\mathbf{A}}\mathbf{x}_e(t) + \hat{\mathbf{B}}\mathbf{u}(t),\ t \neq t_k. \tag{9}$$

During this "open-loop" period, the dynamics of $\hat{P}_K$ is given by

$$\mathbf{z}(t+1) = \Lambda_0 \mathbf{z}(t) \tag{10}$$

where $\mathbf{z}(t) \triangleq [\mathbf{x}^T(t)\ \mathbf{x}_e^T(t)]^T$ and

$$\Lambda_0 = \begin{bmatrix} \mathbf{A} & \mathbf{BK} \\ \mathbf{0} & \hat{\mathbf{A}} + \hat{\mathbf{B}}\mathbf{K} \end{bmatrix}. \tag{11}$$

The controller output $\mathbf{u}(t)$ and the estimator state $\mathbf{x}_e(t)$ generated during this period are stored in the controller.

At $t = t_k$, the estimator receives $\hat{\mathbf{y}}(t_k)$, which includes the consecutive q_k measurements as shown in (4). The newly received $\hat{\mathbf{y}}(t_k)$ as well as the controller's historical information $\mathbf{x}_e(t_k - q_k + 1)$ and $\{\mathbf{u}(t_k), \mathbf{u}(t_k - 1), \cdots, \mathbf{u}(t_k - q_k + 1)\}$ are used to estimate $\mathbf{x}_e(t_k + 1)$. Note that this estimation is calculated within one sampling period T. After the estimation, the historical information are discarded by the controller. In order to demonstrate clearly how the estimate $\mathbf{x}_e(t_k + 1)$ is generated, let us define the *virtual* estimator state $\tilde{\mathbf{x}}(t)$ and the following virtual iterative procedure for updating $\tilde{\mathbf{x}}(t)$ during the virtual period $t \in [t_k - q_k + 1,\ t_k]$. Given the initial condition of the virtual state $\tilde{\mathbf{x}}(t)$

$$\tilde{\mathbf{x}}(t_k - q_k + 1) \triangleq \mathbf{x}_e(t_k - q_k + 1) \tag{12}$$

where $\mathbf{x}_e(t_k - q_k + 1)$ is available from the controller's stored information, for $t \in [t_k - q_k + 1,\ t_k]$,

$$\begin{aligned}\tilde{\mathbf{x}}(t+1) &= \hat{\mathbf{A}}\tilde{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t) + \mathbf{L}\Big(\hat{\mathbf{C}}\tilde{\mathbf{x}}(t) - \mathbf{y}(t)\Big) \\ &= -\mathbf{LC}\mathbf{x}(t) + \hat{\mathbf{B}}\mathbf{K}\mathbf{x}_e(t) + (\hat{\mathbf{A}} + \mathbf{L}\hat{\mathbf{C}})\tilde{\mathbf{x}}(t).\end{aligned} \tag{13}$$

Note that $\mathbf{y}(t)$ for $t \in [t_k - q_k + 1,\ t_k]$ are available from the received $\hat{\mathbf{y}}(t_k)$, and $\mathbf{u}(t)$ for $t \in [t_k - q_k + 1,\ t_k]$ are available from the historical information stored in the controller. The *real* estimator state $\mathbf{x}_e(t_k + 1)$ is simply given by

$$\mathbf{x}_e(t_k + 1) \triangleq \tilde{\mathbf{x}}(t_k + 1). \tag{14}$$

Define $\tilde{\mathbf{z}}(t) \triangleq [\mathbf{x}^T(t)\ \mathbf{x}_e^T(t)\ \tilde{\mathbf{x}}^T(t)]^T$. From (12) and (13), at $t = t_k - q_k + 1$, we derive

$$\tilde{\mathbf{z}}(t_k - q_k + 2) = \Lambda_1 \cdot \mathbf{F}_1 \cdot \mathbf{z}(t_k - q_k + 1) \tag{15}$$

where

$$\Lambda_1 = \begin{bmatrix} \mathbf{A} & \mathbf{BK} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{A}} + \hat{\mathbf{B}}\mathbf{K} & \mathbf{0} \\ -\mathbf{LC} & \hat{\mathbf{B}}\mathbf{K} & \hat{\mathbf{A}} + \mathbf{L}\hat{\mathbf{C}} \end{bmatrix},\ \mathbf{F}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{16}$$

For $t \in [t_k - q_k + 2,\ t_k + 1)$, from (13) and (15), we derive

$$\tilde{\mathbf{z}}(t) = \Lambda_1^{t - t_k + q_k - 1} \cdot \mathbf{F}_1 \cdot \mathbf{z}(t_k - q_k + 1). \tag{17}$$

Noting (13), (14) and (17), we derive the dynamics of $\hat{P}_K$ at $t = t_k$ as

$$\begin{aligned}\mathbf{z}(t_k + 1) &= \Lambda_2 \cdot \mathbf{F}_2 \cdot \tilde{\mathbf{z}}(t_k) \\ &= \Lambda_2 \cdot \mathbf{F}_2 \cdot \Lambda_1^{q_k - 1} \cdot \mathbf{F}_1 \cdot \mathbf{z}(t_k - q_k + 1)\end{aligned} \tag{18}$$

where

$$\Lambda_2 = \begin{bmatrix} \mathbf{A} & \mathbf{BK} \\ -\mathbf{LC} & \hat{\mathbf{A}} + \hat{\mathbf{B}}\mathbf{K} + \mathbf{L}\hat{\mathbf{C}} \end{bmatrix},\ \mathbf{F}_2 = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{19}$$

We are now ready to examine the dynamics of the NCS $\hat{P}_K$ during the interval h_{k+1}. For $t \in (t_k, t_{k+1}]$, from (10), the system response is

$$\mathbf{z}(t) = \Lambda_0^{t - t_k - 1}\mathbf{z}(t_k + 1). \tag{20}$$

Note that we can use the closed-loop dynamics (18) to derive $\mathbf{z}(t_k + 1)$. Using (20) to calculate $\mathbf{z}(t_k - q_k + 1)$ yields

$$\begin{aligned}\mathbf{z}(t_k - q_k + 1) &= \Lambda_0^{t_k - t_{k-1} - q_k}\mathbf{z}(t_{k-1} + 1) \\ &= \Lambda_0^{h_k - q_k}\mathbf{z}(t_{k-1} + 1).\end{aligned} \tag{21}$$

From (18) and (21), we have

$$\mathbf{z}(t_k + 1) = \Lambda_2 \mathbf{F}_2 \Lambda_1^{q_k - 1}\mathbf{F}_1 \Lambda_0^{h_k - q_k}\mathbf{z}(t_{k-1} + 1). \tag{22}$$

Assume that the initial condition is

$$\mathbf{z}_0 = \mathbf{z}(t_0 + 1). \tag{23}$$

In view of (20), the NCS $\hat{P}_K$ can be represented by:

$$\mathbf{z}(t) = \Lambda_0^{t - t_k - 1}\Big(\prod_{j=1}^{k}\mathbf{M}(h_j, q_j)\Big)\mathbf{z}_0,\ t \in (t_k, t_{k+1}] \tag{24}$$

where

$$\mathbf{M}(h_j, q_j) = \Lambda_2 \mathbf{F}_2 \Lambda_1^{q_j - 1}\mathbf{F}_1 \Lambda_0^{h_j - q_j} \tag{25}$$

for all $q_j \in \mathcal{Q}_j$ and $h_j \in \mathcal{N}$.

Let we construct the system, denoted as $\hat{P}_{Ks}$, by sampling $\hat{P}_K$ at the update instants t_k. If we define

$$\bar{\mathbf{z}}(k) \triangleq \mathbf{z}(t_k+1), \tag{26}$$

then $\hat{P}_{Ks}$ is described by

$$\bar{\mathbf{z}}(k+1) = \mathbf{M}(h_k, q_k)\ \bar{\mathbf{z}}(k) \tag{27}$$

with the initial state $\bar{\mathbf{z}}_0$. From (23) and (26), $\bar{\mathbf{z}}_0 = \mathbf{z}_0$.

Remark 1: The switched estimator switches between the two operational modes during each h_k. The open-loop estimator (9) operates from $t_{k-1}+1$ to $t_k - 1$. At t_k, the controller receives $\hat{\mathbf{y}}(t_k)$ and the estimator switches to the close-loop estimator (13), which effectively "operates" from $t_k - q_k + 1$ to t_k. Thus, the switched estimator effectively has the following "sampling" patterns, switching alternatively between the open-loop and close-loop estimators

$$\{(h_1-q_1)T,\ T^{q_1};\ (h_2-q_2)T,\ T^{q_2};\ \cdots\ ;\ (h_k-q_k)T,\ T^{q_k};\ \cdots\}.$$

Since both the update interval h_k and the buffer length q_k are random, the effective "sampling period" for the estimator in (13) is highly time-varying. In this study, the estimator gain matrix $\mathbf{L}$ in (13) is fixed and is designed under the standard condition of constant sampling rate. This is clearly non-optimal, as it cannot match the time-varying characteristics of the underlying system. Design of the optimal time-varying $\mathbf{L}$ to match the underlying time-varying system is a challenging task and is beyond the scope of this study.

IV. Stability of Networked Control System

We analyse the stability properties of NCSs. For the NCS with the arbitrary transmission, the sufficient condition for stability is derived by adopting a common Lyapunov function approach. For the NCS with the Markovian transmission, the sufficient conditions for mean square stability are established by means of the MJLS approach.

A. Lyapunov Stability with Arbitrary Transmission

In the following, $\mathbf{z}(t, \mathbf{z}_0)$ denotes the response of $\hat{P}_K$ (24) with the initial condition $\mathbf{z}_0$, and $\bar{\mathbf{z}}(k, \bar{\mathbf{z}}_0)$ the response of $\hat{P}_{Ks}$ (27) with the initial condition $\bar{\mathbf{z}}_0$.

Definition 1: For given initial condition $\mathbf{z}_0$, $\hat{P}_K$ is Lyapunov asymptotically stable, if for any $\varepsilon > 0$ there exists $\beta > 0$ such that the solution of (24) satisfies

$$\|\mathbf{z}(t,\mathbf{z}_0)\| \le \varepsilon,\ \forall t \ge t_0+1,\ \text{and}\ \lim_{t\to\infty} \|\mathbf{z}(t,\mathbf{z}_0)\| = 0$$

whenever $\|\mathbf{z}_0\| < \beta$.

Theorem 1: The NCS $\hat{P}_K$ is Lyapunov asymptotically stable if there exists a common positive definite matrix $\mathbf{G}_a \in \mathbb{R}^{2n\times 2n}$ such that

$$\mathbf{M}(h_k, q_k)^T \mathbf{G}_a \mathbf{M}(h_k, q_k) - \mathbf{G}_a < \mathbf{0} \tag{28}$$

for all $q_k \in \mathcal{Q}_k$ and $h_k \in \mathcal{N}$.

Proof: For $\hat{P}_{Ks}$, the Lyapunov function is given by

$$V(k) \triangleq \bar{\mathbf{z}}(k)^T \mathbf{G}_a \bar{\mathbf{z}}(k). \tag{29}$$

From (27) and (29) we have

$$V(k+1) = \bar{\mathbf{z}}(k)^T \mathbf{M}(h_k,q_k)^T \mathbf{G}_a \mathbf{M}(h_k,q_k)\bar{\mathbf{z}}(k).$$

Thus, $V(k+1) - V(k) < 0$ if the inequality (28) holds. Hence $\lim_{k\to\infty} \|\bar{\mathbf{z}}(k,\bar{\mathbf{z}}_0)\| = 0$.

The first factor in (24) satisfies

$$\|\Lambda_0^{t-t_k-1}\| \le (\bar{\sigma}(\Lambda_0))^{t-t_k-1} \le (\bar{\sigma}(\Lambda_0))^N \tag{30}$$

where the second inequality holds since $t - t_k - 1$ is always smaller than N. Thus, $\|\Lambda_0^{t-t_k-1}\|$ is always bounded and

$$\lim_{t\to\infty} \|\mathbf{z}(t,\mathbf{z}_0)\| \le (\bar{\sigma}(\Lambda_0))^N \cdot \lim_{k\to\infty} \|\bar{\mathbf{z}}(k,\bar{\mathbf{z}}_0)\| = 0. \tag{31}$$

From Definition 1, the NCS $\hat{P}_K$ is asymptotically stable. ■

Remark 2: For given $\mathbf{K}$ and $\mathbf{L}$, (28) is a set of linear matrix inequalities (LMIs). The LMI tool box [23] can be used to find a feasible solution $\mathbf{G}_a$.

B. Mean Square Stability with Markovian Transmission

For notational convenience, when $h_k = i$ and $q_k = r$, $\mathbf{M}(h_k, q_k)$ is also denoted as $\mathbf{M}(i, r)$.

Definition 2: The NCS $\hat{P}_K$ is mean square stable if $\lim_{t\to\infty} E[\|\mathbf{z}(t,\mathbf{z}_0)\|^2] = 0$ for any initial state $\mathbf{z}_0$.

Theorem 2: The NCS $\hat{P}_K$ with Markovian transmission is mean square stable if there exists a set of matrices $\{\mathbf{G}(i,r) > \mathbf{0}; i \in \mathcal{N}, r \in \mathcal{Q}_k\}$ satisfying

$$\mathbf{M}(i,r)^T \bar{\mathbf{G}}_i \mathbf{M}(i,r) - \mathbf{G}(i,r) < \mathbf{0} \tag{32}$$

for all $i \in \mathcal{N}$ and $r \in \mathcal{Q}_k$, where

$$\begin{aligned} \bar{\mathbf{G}}_i &\triangleq E\left[\mathbf{G}(h_{k+1}, q_{k+1})\ |_{h_k=i,q_k=r}\right] \\ &= \sum_{j=1}^{N} p_{ij} \sum_{r=1}^{s_{k+1}} \frac{1}{s_{k+1}} \cdot \mathbf{G}(j,r). \end{aligned} \tag{33}$$

Proof: Since $h_k \in \mathcal{N}$ is driven by a discrete-time Markov chain, $\hat{P}_{Ks}$ is in fact a discrete-time MJLS with N operation modes [14]. For the system $\hat{P}_{Ks}$ of (27), consider the stochastic Lyapunov function which is given by:

$$V(\bar{\mathbf{z}}(k), h_k, q_k) \triangleq \bar{\mathbf{z}}(k)^T \mathbf{G}(h_k, q_k)\bar{\mathbf{z}}(k).$$

Noticing (6), (7) and (27), we have

$$\begin{aligned} &E[\Delta V(\bar{\mathbf{z}}(k), h_k, q_k)] = \\ &E\left[\bar{\mathbf{z}}(k+1)^T \mathbf{G}(h_{k+1}, q_{k+1})\bar{\mathbf{z}}(k+1)\ |_{\bar{\mathbf{z}}(k),h_k=i,q_k=r}\right] \\ &-\bar{\mathbf{z}}(k)^T \mathbf{G}(i,r)\bar{\mathbf{z}}(k) = \\ &E\left[\bar{\mathbf{z}}(k)^T \mathbf{M}(i,r)^T \mathbf{G}(h_{k+1}, q_{k+1})\mathbf{M}(i,r)\bar{\mathbf{z}}(k)\ |_{h_k=i,q_k=r}\right] \\ &-\bar{\mathbf{z}}(k)^T \mathbf{G}(i,r)\bar{\mathbf{z}}(k) = \\ &\bar{\mathbf{z}}(k)^T \mathbf{M}(i,r)^T \bar{\mathbf{G}}_i \mathbf{M}(i,r)\bar{\mathbf{z}}(k) - \bar{\mathbf{z}}(k)^T \mathbf{G}(i,r)\bar{\mathbf{z}}(k) = \\ &\bar{\mathbf{z}}(k)^T \big(\mathbf{M}(i,r)^T \bar{\mathbf{G}}_i \mathbf{M}(i,r) - \mathbf{G}(i,r)\big)\bar{\mathbf{z}}(k). \end{aligned} \tag{34}$$

If (32) is met, from (34) we have

$$E[\Delta V(\bar{\mathbf{z}}(k), h_k, q_k)] < 0$$

which implies that

$$\lim_{k\to\infty} E[\|\bar{\mathbf{z}}(k,\bar{\mathbf{z}}_0)\|^2] = 0.$$

TABLE I

THE POSITIVE DEFINITE MATRIX $\mathbf{G}_a$ CHOSEN FOR THE ARBITRARY TRANSMISSION CASE.

$$\mathbf{G}_a = \begin{bmatrix} 0.5307 & -0.2359 & 0.0330 & -0.3215 & 0.0706 & -0.0276 \\ -0.2359 & 0.2308 & -0.0182 & 0.0201 & -0.0368 & 0.0348 \\ 0.0330 & -0.0182 & 0.0570 & 0.0360 & -0.0081 & -0.0227 \\ -0.3215 & 0.0201 & 0.0360 & 0.6772 & -0.2495 & -0.0313 \\ 0.0706 & -0.0368 & -0.0081 & -0.2495 & 0.1970 & 0.0317 \\ -0.0276 & 0.0348 & -0.0227 & -0.0313 & 0.0317 & 0.0534 \end{bmatrix}$$

From Definition 2, $\hat{P}_{Ks}$ is mean square stable. Noting (30) again, we have

$$\lim_{t\to\infty} E[\|\mathbf{z}(t,\mathbf{z}_0)\|^2] \le (\bar{\sigma}(\Lambda_0))^N \cdot \lim_{k\to\infty} E[\|\bar{\mathbf{z}}(k,\bar{\mathbf{z}}_0)\|^2] = 0 \tag{35}$$

for any initial state $\mathbf{z}_0$. This completes the proof. ■

V. A NUMERICAL EXAMPLE

We considered the following unstable system of $\mathbf{x}(t) \in \mathbb{R}^3$ with the true plant matrices given by

$$\mathbf{A} = \begin{bmatrix} -1.05 & 0 & 0 \\ -2 & 0.75 & 0 \\ 0 & 1.05 & 0.5 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.$$

The matrices of the plant model were however given by

$$\hat{\mathbf{A}} = \begin{bmatrix} -1.07 & -0.01 & 0.03 \\ -1.99 & 0.76 & 0.02 \\ 0.01 & 1.04 & 0.51 \end{bmatrix},\ \hat{\mathbf{B}} = \begin{bmatrix} 0.49 \\ 0.05 \\ 0.52 \end{bmatrix},$$

$$\hat{\mathbf{C}} = \begin{bmatrix} 0.98 & 1.01 & 0.02 \end{bmatrix}.$$

The controller was designed to have the state feedback gain matrix and estimator gain matrix of

$$\mathbf{K} = \begin{bmatrix} -0.27 & 0.57 & 0.02 \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} 0.26 & 0.14 & -0.17 \end{bmatrix}^T,$$

respectively, for the NCS $\hat{P}_K$. The initial state was chosen to be $\mathbf{z}_0 = [-2\ 0\ 2\ -1\ 0\ 1]^T$. The value of q_{max} was set to $q_{max} = 5$, and the value of N to $N = 8$.

In the general case of arbitrary transmission, no particular probability distribution was imposed on h_k, and h_k arbitrarily took values from the finite set $\mathcal{N} \triangleq \{1,\ldots,8\}$. Choosing the positive definite matrix $\mathbf{G}_a \in \mathbb{R}^{6\times 6}$ as given in Table 1, it was easily verified that the 30 inequalities in (28) were satisfied. According to Theorem 1, the NCS $\hat{P}_K$ was Lyapunov stable. Fig. 3 depicts the state trajectories of the plant $\hat{P}$, $\mathbf{x}(t)$, and the estimator, $\mathbf{x}_e(t)$, for the given sequences of the update instants and the buffer lengths shown in Fig. 3 (c).

The NCS with the Markovian transmission case was next investigated, where h_k took values from the finite set $\mathcal{N} \triangleq \{1,\ldots,8\}$ with the transition probability matrix given by

$$\Gamma = \begin{bmatrix} 0.4 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 & 0 & 0 \\ 0.2 & 0.3 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0 \\ 0.1 & 0.2 & 0.3 & 0.1 & 0.1 & 0.1 & 0.1 & 0 \\ 0.1 & 0.1 & 0.2 & 0.3 & 0.1 & 0.1 & 0.1 & 0 \\ 0 & 0.1 & 0.1 & 0.1 & 0.3 & 0.2 & 0.1 & 0.1 \\ 0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.3 & 0.2 & 0.1 \\ 0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.3 & 0.2 \\ 0 & 0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.2 & 0.4 \end{bmatrix}.$$

The bursty nature of the network was well modelled by this Γ with $p_{ii} > p_{ij}$, for all $i,j \in \mathcal{N}$ and $j \neq i$, which means that the likelihood of long update interval after a long update interval transmission is higher than after a short update interval transmission. A set of the matrices $\{\mathbf{G}(i,r)\}$ were found, which ensured that all the 30 inequalities in (32) were satisfied. According to Theorem 2, the NCS $\hat{P}_K$ was mean square stable. This set of $\{\mathbf{G}(i,r)\}$ was not included here owing to space limitation. A typical response of the NCS $\hat{P}_K$ with the initial condition $h_0 = 5$ are shown in Fig 4. The system was simulated 200 times with the equal initial state probabilities Prob$(h_0 = i)$ for all $i \in \mathcal{N}$. Fig. 5 depicted $E[\|\mathbf{z}(t,\mathbf{z}_0)\|^2]$, calculated by averaging over the 200 simulations. The result of Fig. 5 indicates that $\lim_{t\to\infty} E[\|\mathbf{z}(t,\mathbf{z}_0)\|^2] = 0$, which confirmed that the NCS was mean square stable.

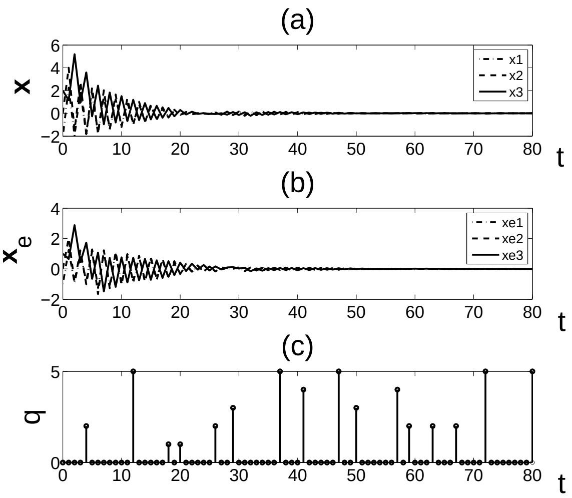

Fig. 3. Simulation result for arbitrary transmission case: (a) state trajectories of the plant $\hat{P}$, (b) state trajectories of the estimator, and (c) sequences of the update instants $\{t_k\}$ and the buffer lengths $\{q_k\}$.

VI. CONCLUSIONS

Stability properties have been analysed for the discrete-time networked control systems with stochastic update in-

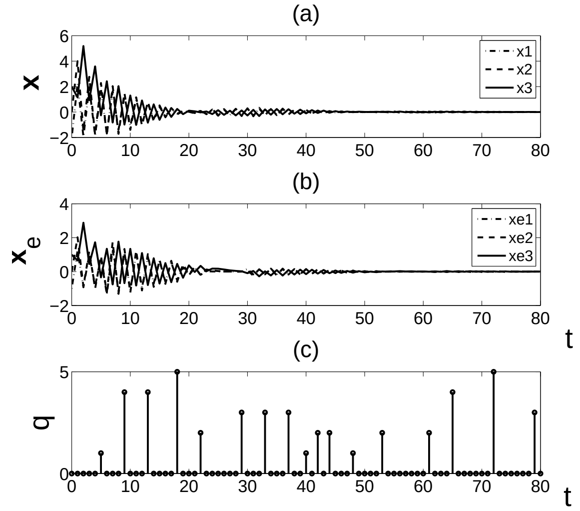

Fig. 4. Simulation result for Markovian transmission case: (a) state trajectories of the plant $\hat{P}$, (b) state trajectories of the estimator, and (c) sequences of the update instants $\{t_k\}$ and the buffer lengths $\{q_k\}$.

tervals and random buffer lengths. For the generic case of arbitrary transmission where no specific probability distribution is imposed on the sequence of successful transmission instants, the sufficient condition for stability has been derived by adopting a common Lyapunov function approach. For the case of Markovian transmission where the sequence of update intervals is driven by a Markov chain, the sufficient conditions for mean square stability have been established by employing the Markovian jump linear system approach. The effectiveness of the proposed stability analysis approach for NCSs has been demonstrated with a numerical example.

ACKNOWLEDGEMENTS

This work was supported by the National Natural Science Foundation of China (Grants No.60774001, No.60736021 and No.60721062), 973 program of China (Grant No.2002CB312200), program for New Century Excellent Talents in University (NCET-04-0547) and the 111 Project (Grant No.B07031).

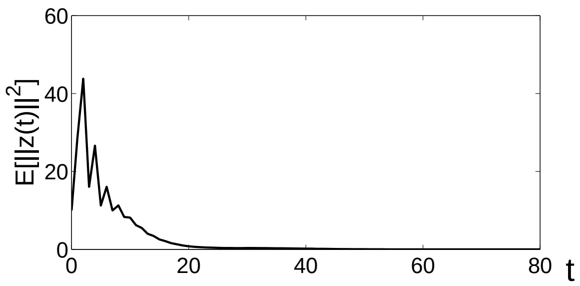

Fig. 5. Confirmation of mean square stability of the NCS: $\lim_{t\to\infty} E[\|\mathbf{z}(t,\mathbf{z}_0)\|^2] = 0$, where $E[\|\mathbf{z}(t,\mathbf{z}_0)\|^2]$ was calculated by averaging over 200 simulation runs.

REFERENCES

[1] Y. Tipsuwan and M.Y. Chow, "Control methodologies in networked control systems," *Control Eng. Practice*, vol.11, no.10, pp.1099–1111, 2003.

[2] P. Antsaklis and J. Baillieul, "Guest editorial: Special issue on networked control systems," *IEEE Trans. Automatic Control*, vol.49, no.9, pp.1421–1423, 2004.

[3] J.P. Hespanha, P. Naghshtabrizi and Y.-G. Xu, "A survey of recent results in networked control systems," *Proc. IEEE*, vol.95, no.1, pp.138–162, 2007.

[4] G.C. Walsh, H. Ye and L. Bushnell, "Stability analysis of networked control systems," *Proc. 1999 American Control Conf.*, June 2-4, 1999, Vol.4, pp.2876–2880.

[5] W. Zhang, M.S. Branicky and S.M. Phillips, "Stability of networked control systems," *IEEE Control System Magazine*, vol.21, no.1, pp.84–99, 2001.

[6] L.A. Montestruque and P. Antsaklis, "Stability of model-based networked control systems with time-varying transmission times," *IEEE Trans. Automatic Control*, vol.49, no.9, pp.1562–1572, 2004.

[7] M. García-Rivera and A. Barreiro, "Analysis of networked control systems with drops and variable delays," *Automatica*, vol.43, no.12, pp.2054–2059, 2007.

[8] M. Tabbara, D. Nesic and A.R. Teel, "Stability of wireless and wireline networked control systems," *IEEE Trans. Automatic Control*, vol.52, no.9, pp.1615–1630, 2007.

[9] J. Nilsson, B. Bernhardsson and B. Wittenmark, "Stochastic analysis and control of real-time systems with random time delays," *Automatica*, vol.34, no.1, pp.57–64, 1998.

[10] P.V. Zhivoglyadov and R.H. Middleton, "Networked control design for linear systems," *Automatica*, vol.39, no.4, pp.743–750, 2003.

[11] P. Seiler and R. Sengupta, "An H_∞ approach to networked control," *IEEE Trans. Automatic Control*, vol.50, no.3, pp.356–364, 2005.

[12] J. Xiong and J. Lam, "Stabilization of linear systems over networks with bounded packet loss" *Automatica*, vol.43, no.1, pp.80–87, 2007.

[13] J. Wu and T. Chen, "Design of networked control systems with packet dropouts," *IEEE Trans. Automatic Control*, vol.52, no.7, pp.1314–1319, 2007.

[14] Y. Ji, H.J. Chizeck, X. Feng and K.A. Loparo, "Stability and control of discrete-time jump linear systems," *Control Theory Adv. Technol.*, vol.7, no.2, pp. 247–270, 1991.

[15] O.L.V. Costa and M.D. Fragoso, "Stability results for discrete-time linear systems with Markovian jumping parameters," *J. Math. Analysis and Appl.*, vol.179, no.1, pp.154–178, 1993.

[16] J.R. Moyne and D.M. Tilbury, "The emergence of industrial control networks for manufacturing control, diagnostics, and safety data," *Proc. IEEE*, vol.95, no.1, pp.29–47, 2007.

[17] IEEE Std 802.3 Part 3: *Carrier Sense Multiple Access with Collision Detection (CSMA/CD) Access Method and Physical Layer Specifications*. IEEE, 2005.

[18] IEEE Std 802.11 Part 11: *Wireless LAN Medium Access Control (MAC) and Physical Layer (PHY) Specifications*. IEEE, 2007.

[19] G.P. Liu, Y. Xia, J. Chen, D. Rees and W. Hu, "Networked predictive control of systems with random network delays in both forward and feedback channels" *IEEE Trans. Industrial Electronics*, vol.54, no.3, pp.1282–1297, 2007.

[20] D.E. Quevedo, E.I. Silva and G.C. Goodwin, "Packetized predictive control over erasure channels," *Proc. 2007 American Control Conf.*, July 11-13, 2007, pp.1003–1008.

[21] Y.-L. Wang and G.-H. Yang, "H_∞ control of networked control systems with delay and packet disordering via predictive method," *Proc. 2007 American Control Conf.*, July 11-13, 2007, pp.1021–1026.

[22] C.L. Robinson and P.R. Kumar, "Sending the most recent observation is not optimal in networked control: Linear temporal coding and towards the design of a control specific transport protocol," *Proc. of CDC 2007*, Dec.12-14, 2007, pp.334–339.

[23] S. Boyd, L.E. Ghaoui, E. Feron and V. Balakrishnan, "Linear matrix inequalities in systems and control theory," in *Studies in Applied Mathematics*. Philadelphia, PA: SIAM, 1994.