$$
\begin{aligned}
&\cdot \left(\frac{2l_{11}^4 k_2^2 x_2^2 (1+z_2^2)^{1/2}}{9\delta_2^2 d_2^2 \beta_2^4} + \frac{2l_{11}^4 k_2^2 z_2^2}{9\delta_2^2 d_2^2 \beta_2^4} (1+x_2^2)^{1/2} \cdot \right. \\
&\left. \left. + \frac{4l_{11}^6 k_2^3 z_2^2 x_2^2}{27\delta_2^3 d_2^3 \beta_2^{7/3}} (1+x_2^2)^{1/2} \right) \right) \left(d_2 (1+x_3^6)^{1/2} + a_2\right) \\
&\left. + \frac{l_{11}^3 a_1 k_2}{3\delta_2 d_2 \beta_2^2} (1+x_2^2)^{1/2} \right) \\
&\gamma_{413}(\cdot) = \beta_3 + \frac{k_2^2 d_2^2 z_3^2}{2\varepsilon_{20} k_3} + \frac{3125}{32k_3} \varepsilon_{21}^{-5} k_2^6 d_2^6 z_3^2 \beta_2^{12} + 1.8899 d_2 \\
&\cdot \delta_{31}^{-1/3} \gamma_{31}^{1/3} \left(\beta_2 + \left(1 + \frac{l_{11}^4 k_2^2 z_2^2 x_2^2}{9\delta_2^2 d_2^2 \beta_2^4} \right)^{1/2} \right) \\
&\cdot (1+x_3^6)^{-1/2} (1+z_3^2 x_3^{10})^{1/2} \\
&\gamma_{414}(\cdot) = l_{11}\beta_2\beta_3 + \frac{1}{2\varepsilon_{20} k_3} l_{11} k_2^2 d_2^2 z_3^2 \beta_2 + \frac{3125}{32k_3} l_{11} \varepsilon_{21}^{-5} k_2^6 \\
&\cdot d_2^6 z_3^2 \beta_2^{13} + \frac{0.07 l_{11}^5 k_2^2 x_2^2 \gamma_{31}^{1/3}}{\delta_2^2 \delta_{31}^{1/3} d_2^2 \beta_2^4} \left(1 + \frac{l_{11}^4 k_2^2 z_2^2 x_2^2}{9\delta_2^2 d_2^2 \beta_2^4} \right)^{-1/2} \\
&\cdot \left(d_2 (1+x_3^6)^{1/2} + a_2 \right) (1+z_2^2 z_3^2)^{1/2}
\end{aligned}
$$

and $z_{j2} = c_2 + \rho_{20} + \rho_{21}$, $\delta_4 = \delta_3 + \delta_{41} + \delta_{42}$, δ_{41}, δ_{42}, $c_i > \varepsilon_i > 0$ $(i = 1, 2, 3)$, $c_4 > 0$.

In simulations, we choose $y_r(t) = 0.2 \sin t$ whose physical unit is rad, parameters $c_1 = 2.001$, $c_2 = 1.501$, $c_3 = 2.01$, $c_4 = 0.001$, $\varepsilon_1 = 2$, $\varepsilon_{20} = 0.3$, $\varepsilon_{21} = 1.2$, $\varepsilon_3 = 2$, $k_1 = 2$, $k_2 = 0.5$, $k_3 = 10$, $k_4 = 100$, $\delta_2 = 0.1$, $\delta_{31} = 0.1$, $\delta_{41} = 0.1$, $\delta_{42} = 0.1$, and the initial values $x_1(0) = \theta(0) = 0.1$, $x_2(0) = 0.1$, $x_3(0) = 0$, $x_4(0) = 0.1$ whose units are $\mathrm{rad}, \mathrm{rad/s}, m, \mathrm{m/s}$. Fig. 2 demonstrates the effectiveness of the control scheme, where the unit of y axis for the tracking error z_1 is rad, the unit of y axis for the control input v is N.

V. Conclusions

This note considers the output tracking of high-order stochastic nonlinear systems without imposing any restriction on the high-order and the drift and diffusion terms, and apply the control scheme to stochastic benchmark mechanical system.

There are two remaining problems to be investigated: 1) an issue is whether there exists a smooth or continuous feedback controller to guarantee that the equilibrium at the origin of the closed-loop system is globally asymptotically stable in probability without assumptions H1) and/or H2) and 2) how to design the controller for this practical example in which k and k_s are dealt with as stochastic noise simultaneously.

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Stability of Networked Control Systems With Polytopic Uncertainty and Buffer Constraint

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***Abstract*—We consider the stability of the discrete-time networked control systems (NCSs) with polytopic uncertainty, where a smart controller is updated with the buffered sensor information at stochastic intervals and the amount of the buffered data received by the controller under the buffer capacity constraint is also random. We establish sufficient conditions for guaranteeing the exponential stability of generic switched NCSs and the exponential mean square stability of Markov-chain driven NCSs, respectively. An illustrative example is given to demonstrate the effectiveness of our results.**

***Index Terms*—Buffer constraint, networked control systems (NCSs), polytopic uncertainty, stochastic systems.**

I. Introduction

Networked control systems (NCSs) have received much attention recently [1]–[3]. An NCS is a control system in which a control loop is closed via a shared communication network. The use of a shared network in the feedback path offers the advantages of low installation cost, reducing system wiring, simple system diagnosis and easy maintenance. However, the NCS also has some inherent shortcomings, such

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as bandwidth constraints, packet delays and packet dropouts, which will degrade system performance or even cause closed-loop instability. Hence, guaranteeing closed-loop stability is an upmost requirement for any NCS. Stability analysis of NCSs is investigated in [4]–[6], and stabilizing controllers are designed in [7]–[12]. Stochastic approaches based on the mean-square stability [13], [14] are typically adopted to deal with network delay and packet dropout. Under such a stochastic approach, the network is modelled as a Markov process and the system is a discrete-time Markovian jump linear system [9].

A feature of modern network protocols is that data is sent in large packets. This opens up the possibility to conceive control schemes in which a large amount of data, consisting of the current and historical values, are sent through the network at the transmission instant. The works [15] and [16] adopt packetized predictive controllers, which pre-compute control data valid on a given time horizon and then transmit the data in a single packet to the actuator at the next network access, in order to maintain the performance of the NCS with a parsimonious access of the network. The controller in [17], which has a two-way communication with a wireless sensor node, sends to the sensor a sequence of state predictions at every suitable instant to reduce the sensor transmission load. For the NCS with the network separating the sensor and controller, the work [18] shows that sending a linear combination of the past and present two measurements with appropriate weightings is better than sending the most recent observation only. In the study [7], a buffer is used to store sensor measurements on the plant at instants between two consecutive accesses to the network. The buffered data are transmitted to the observer-based controller at each access instant. The results of [7] show that the stability of a uncertainty-free NCS can always be achieved, provided that the length of each packet is no less than the plant observability index, and the interval between two consecutive successful transmissions is bounded. Thus, the technique of [7] cannot be applied to the NCS where the sensor buffer capacity is smaller than the plant observability index or the NCS with uncertainty. In practice, a small buffer capacity is preferred, owing to the cost of sensor devices and/or the limited power consumption in wireless sensor networks.

To the best of our knowledge, no work to date addresses this practical problem of buffer capacity constraint. The novelty of this contribution is that we analyse the stability of the NCS with polytopic uncertainty under this limited buffer capacity. In our NCS, the communication channel between the sensor/buffer and the controller is subject to random access delays and packet dropouts, while the channel between the controller and the actuator has a guaranteed bandwidth which is equivalent to a dedicated link. Note that such a network configuration is practical, since modern communication protocols are extremely flexible and are capable of offering different quality of services. Our model is not the generic NCS model but nevertheless it represents an important class of NCSs. Many important NCSs, such as the work of [7], are developed based on this model. Since the controller receives data at random intervals and the amount of data received each time is also random, the NCS is driven by an underlying discrete-time stochastic process. Moreover, we do not assume a uncertainty-free NCS and our model is subject to polytopic uncertainty. We establish sufficient conditions for ensuring the exponential stability of generic switched NCSs and the exponential mean square stability of Markov-chain driven NCSs, respectively.

The following notational conventions are adopted. $\mathbb{R}$ stands for real numbers and $\mathbb{N}$ for nonnegative integers. For vector $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\| \triangleq \sqrt{\mathbf{x}^\mathsf{T}\mathbf{x}}$. $\mathbf{W} > \mathbf{0}$ indicates that $\mathbf{W}$ is a positive-definite matrix, while $\mathbf{I}$ and $\mathbf{0}$ represent the identity and zero matrices of appropriate dimension, respectively. Finally $E[\mathbf{y}(t)]$ defines the expectation of $\mathbf{y}(t)$.

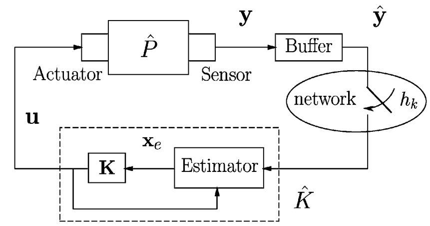

Fig. 1. Networked control system $\hat{P}_K$.

II. Description of NCSs

The NCS $\hat{P}_K$ of Fig. 1 contains a discrete-time plant $\hat{P}$ and a discrete-time controller $\hat{K}$ with the control loop closed via a shared communication network. The plant $\hat{P}$ is described by

$$\begin{cases} \mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \quad t \in \mathbb{N} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$ and $\mathbf{y}(t) \in \mathbb{R}^p$ are the state, input, and output vectors of the plant, respectively, while $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{p\times n}$ are the true matrices of the plant state-space equation. The channel between the sensor and the controller is subject to random access delay and packet dropout, while the controller/actuator channel is equivalent to a dedicated link. $\hat{P}$ and $\hat{K}$ are time-driven and synchronized. At each $t \in \mathbb{N}$, the sensor takes the measurement of the plant output $\mathbf{y}(t)$ and stores it in the buffer. If the buffer reaches its maximum capacity M, the oldest data in the buffer is discarded to give room for the new data. We denote all the plant outputs stored in the buffer at $t \in \mathbb{N}$ as $\hat{\mathbf{y}}(t)$. At each $t \in \mathbb{N}$, the buffer seeks for the access to transmit $\hat{\mathbf{y}}(t)$ to $\hat{K}$ through the network. If the access is granted, transmission takes place and after the transmission, $\hat{\mathbf{y}}(t)$ is discarded by the buffer. If no access is granted, $\hat{\mathbf{y}}(t)$ is kept in the buffer. Since the waiting time at the buffer node is stochastic, the length of $\hat{\mathbf{y}}(t)$ at transmission time is also random.

There are two alternative outcomes of each transmission: one is that the transmission succeeds and $\hat{K}$ receives $\hat{\mathbf{y}}(t)$ at t; the other is that the transmission fails due to a packet dropout and $\hat{K}$ misses $\hat{\mathbf{y}}(t)$. The packet transmission delay through the network for a successful transmission is assumed to be negligible. Hence, we only consider the packet dropout and the network access delay at the buffer node. Those instants at which transmissions succeed are denoted as t_k, $k \in \mathbb{N}$, in ascending order, and $t_0 = -1$ is assumed without the loss of generality. As $\hat{K}$ receives the new information $\hat{\mathbf{y}}(t_k)$ at t_k, t_k is referred to as the update instant. Define the update interval

$$h_k \triangleq t_{k+1} - t_k, \quad k \in \mathbb{N} \tag{2}$$

which can take values from a finite integer set $h_k \in \mathcal{N} \triangleq \{1, \ldots, N\}$ with the maximal update interval N determined by the busy status of the network. Since the buffer will not be uncleared longer than N steps, the length of data stored in the buffer is never larger than N. Therefore, we assume $M \leq N$. Denote the number of plant output measurements stored in the buffer as q_k. Then $\hat{\mathbf{y}}(t_{k+1})$ is expressed at t_{k+1} as

$$\hat{\mathbf{y}}(t_{k+1}) = \{\mathbf{y}(t_{k+1}), \mathbf{y}(t_{k+1}-1), \cdots, \mathbf{y}(t_{k+1}-q_k+1)\}. \tag{3}$$

Obviously, $q_k \in \mathcal{M} \triangleq \{1, \ldots, M\}$ and $q_k \leq h_k$. A large h_k and a small q_k indicate that at t_{k+1} the controller receives a small amount of update information after a long delay. By contrast, a small h_k and a large q_k show that the controller receives a large amount of update

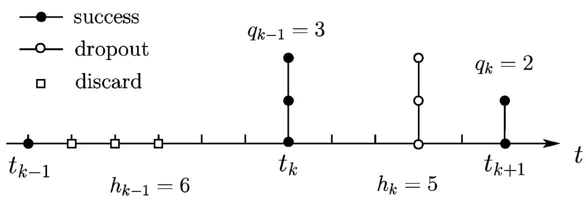

Fig. 2. Time diagram of NCS with buffer data, assuming $M=3, h_{k-1}=6$ and $h_k=5$. Data $\mathbf{y}(t_{k-1}+1), \mathbf{y}(t_{k-1}+2)$ and $\mathbf{y}(t_{k-1}+3)$ are discarded due to the buffer reaching its maximum capacity at $t_{k-1}+4, t_{k-1}+5$, and t_k, respectively. $\{\mathbf{y}(t_k), \mathbf{y}(t_k-1), \mathbf{y}(t_k-2)\}$ are successfully transmitted at t_k and hence $q_{k-1}=3$. At t_k+3 the transmission fails due to packet dropout. $\{\mathbf{y}(t_{k+1}), \mathbf{y}(t_{k+1}-1)\}$ are successfully transmitted at t_{k+1} and hence $q_k=2$.

information quickly. Hence the pair (h_k, q_k) is a measure of the *update quality*. Various scenarios of the NCS with buffer are illustrated in Fig. 2.

It can be seen that for $t \neq t_{k+1}$ the feedback loop is broken and $\hat{P}_K$ is in the mode of open loop, while for $t=t_{k+1}$ $\hat{P}_K$ is in the mode of closed loop. A smart control mechanism $\hat{K}$, similar to the one in [7], is adopted as

$$\mathbf{u}(t)=\mathbf{K}\mathbf{x}_e(t), \quad t \in \mathbb{N} \tag{4}$$

$$\mathbf{x}_e(t+1)=\hat{\mathbf{A}}\mathbf{x}_e(t)+\mathbf{B}\mathbf{u}(t), \quad t \neq t_{k+1} \tag{5}$$

$$\begin{aligned}\mathbf{x}_e(t+1)=&(\hat{\mathbf{A}}+\mathbf{LC})^q\mathbf{x}_e(t-q+1)\\&+\sum_{m=0}^{q-1}(\hat{\mathbf{A}}+\mathbf{LC})^m\left(\mathbf{B}\mathbf{u}(t-m)-\mathbf{L}\mathbf{y}(t-m)\right)\\&t=t_{k+1},\ q=q_k\end{aligned} \tag{6}$$

where $\mathbf{x}_e(t) \in \mathbb{R}^n$ is the estimator state, $\mathbf{K} \in \mathbb{R}^{m \times n}$ and $\mathbf{L} \in \mathbb{R}^{n \times p}$ are the state feedback and estimator gain matrices, respectively, while $\hat{\mathbf{A}} \in \mathbb{R}^{n \times n}$ is the estimate of the plant dynamics matrix $\mathbf{A}$ which is unavailable. Thus, the uncertainty is associated with $\mathbf{A}$, while the plant matrices $\mathbf{B}$ and $\mathbf{C}$ are assumed to be known. For $t \neq t_{k+1}$, the estimator is based on the plant model (5). The controller output $\mathbf{u}(t)$ and the estimator state $\mathbf{x}_e(t)$, generated during this open-loop period, are stored in the controller. At $t=t_{k+1}$, the estimator receives $\hat{\mathbf{y}}(t_{k+1})$, which includes the q_k consecutive plant outputs as shown in (3). The newly received $\hat{\mathbf{y}}(t_{k+1})$ as well as the controller's historical information $\mathbf{x}_e(t_{k+1}-q_k+1)$ and $\{\mathbf{u}(t_{k+1}), \mathbf{u}(t_{k+1}-1), \cdots, \mathbf{u}(t_{k+1}-q_k+1)\}$ are used to estimate $\mathbf{x}_e(t_{k+1}+1)$ by the observer structure (6). After the estimation, the historical information are discarded.

For the given N and M, define the set $\mathcal{S}_{N,M} \triangleq \{(h,q)|h \in \mathcal{N}, q \in \mathcal{M}, q \leq h\}$. The number of elements in the set $\mathcal{S}_{N,M}$ is $\bar{r}=(1/2)(2N-M+1)M$. Next define the set $\mathcal{R}_{NM} \triangleq \{1,2,\cdots,\bar{r}\}$ and the mapping f from $\mathcal{S}_{N,M}$ to $\mathcal{R}_{NM}$

$$\begin{aligned}r&=f(h,q)\\&=\begin{cases}\frac{1}{2}h(h-1)+q, & h \leq M\\ \frac{1}{2}M(M-1)+M(h-M)+q, & h > M.\end{cases}\end{aligned} \tag{7}$$

It is easy to show that f is a one to one mapping. Define its inverse mapping f^{-1}

$$(h,q)=f^{-1}(r)=(g_h(r), g_q(r)) \tag{8}$$

which can be realized by the following iteration algorithm:

- *Step 1*: If $r \leq (1/2)M(M-1)$, go to *Step 2*; otherwise, go to *Step 3*.
- *Step 2*: Find $h \in \mathcal{N}$ to satisfy $(1/2)h(h-1) < r \leq (1/2)h(h+1)$. Then $q=r-(1/2)h(h-1)$. *End*.
- *Step 3*: Find $h \in \mathcal{N}$ to satisfy $(1/2)M(2h-M-1) < r \leq (1/2)M(2h-M+1)$. Then $q=r-(1/2)M(2h-M-1)$. *End*.

Therefore, the sequence $[h_k, q_k]$ can be mapped as $[r_k]$ by f and r_k can also be viewed as a measure of update quality. Define the set $\mathcal{W}_{NM} \triangleq \{[r_0, r_1, \cdots]|r_k \in \mathcal{R}_{NM}, \forall k \in \mathbb{N}\}$ and the state of the NCS as

$$\mathbf{z}(t) \triangleq \left(\mathbf{x}^{\mathsf{T}}(t)\,\mathbf{x}_e^{\mathsf{T}}(t)\right)^{\mathsf{T}}. \tag{9}$$

Definition 1: The NCS $\hat{P}_K$ is exponentially stable (ES) if there exist constants $0<\alpha<1$ and $\beta>1$ such that $\forall [r_k] \in \mathcal{W}_{NM}$, $\forall \mathbf{z}(0) \in \mathbb{R}^{2n}$ and $\forall t \in \mathbb{N}$, $\|\mathbf{z}(t)\|^2 < \beta\alpha^t\|\mathbf{z}(0)\|^2$.

For the above definition, $[r_k]$ is a generic sequence and the NCS is a switched system. Next consider the case where r_k is a stochastic process with a specific probability distribution.

Assumption 1: The stochastic process r_k is driven by a Markov chain and takes values in $\mathcal{R}_{NM}$ with a given transition probability matrix $\mathbf{\Upsilon} \triangleq (p_{ij}) \in \mathbb{R}^{\bar{r} \times \bar{r}}$, where $p_{ij}=\text{Prob}(r_{k+1}=j|r_k=i)$ which are subject to the restrictions $p_{ij} \geq 0$ and $\sum_{j=1}^{\bar{r}} p_{ij}=1$, $\forall i,j \in \mathcal{R}_{NM}$.

Definition 2: (See [14]): The NCS $\hat{P}_K$ under Assumption 1 is exponentially mean square stable (EMSS) if there exist constants $0<\alpha<1$ and $\beta>1$ such that $\forall \mathbf{z}(0) \in \mathbb{R}^{2n}$, $\forall r_0 \in \mathcal{R}_{NM}$ and $\forall t \in \mathbb{N}$, $E[\|\mathbf{z}(t)\|^2|_{\mathbf{z}(0),r_0}] < \beta\alpha^t\|\mathbf{z}(0)\|^2$.

We considers the case that $\mathbf{A} \in \mathbb{R}^{n \times n}$ is subject to the polytopic uncertainty, namely, $\mathbf{A} \in \mathcal{A}$ and

$$\mathcal{A}=\left\{\mathbf{A}|\mathbf{A}=\sum_{j=1}^{l}\omega_j\mathbf{A}_j, \sum_{j=1}^{l}\omega_j=1, \omega_j \geq 0\right\} \tag{10}$$

where $\mathbf{A}_j \in \mathbb{R}^{n \times n}$ with $j \in \{1,\cdots,l\}$ are the l constant matrices that determine the convex polytope. In the next section, we derive the ES and EMSS criteria, respectively, for $\hat{P}_K$ under the polytopic uncertainty of $\mathbf{A} \in \mathcal{A}$.

III. Stability Conditions

At $t=t_{k+1}$, $\hat{P}_K$ is in the mode of closed loop. In order to demonstrate clearly how the estimate $\mathbf{x}_e(t_{k+1}+1)$ of (6) is generated, let us define a *virtual* observer state $\tilde{\mathbf{x}}(t)$ and the following virtual iterative procedure for the observation period $t \in \{t_{k+1}-q_k+1, \cdots, t_{k+1}\}$. At the beginning of each observation period, the initial virtual observer state is $\tilde{\mathbf{x}}(t_{k+1}-q_k+1)=\mathbf{x}_e(t_{k+1}-q_k+1)$. The virtual states are produced by a standard observation law

$$\begin{aligned}\tilde{\mathbf{x}}(t+1)&=\hat{\mathbf{A}}\tilde{\mathbf{x}}(t)+\mathbf{B}\mathbf{u}(t)+\mathbf{L}\left(\mathbf{C}\tilde{\mathbf{x}}(t)-\mathbf{y}(t)\right)\\&=-\mathbf{LC}\mathbf{x}(t)+\mathbf{BK}\mathbf{x}_e(t)+(\hat{\mathbf{A}}+\mathbf{LC})\tilde{\mathbf{x}}(t)\\&\quad t \in \{t_{k+1}-q_k+1,\cdots,t_{k+1}\}.\end{aligned} \tag{11}$$

At the end of this observation period, the estimator state $\mathbf{x}_e(t)$ is given by $\mathbf{x}_e(t_{k+1}+1)=\tilde{\mathbf{x}}(t_{k+1}+1)$, namely, (6). Motivated by the above virtual recursive process, we construct the augmented system $\hat{G}_K$ for $\hat{P}_K$

$$\tilde{\mathbf{z}}(t+1)=\tilde{\mathbf{A}}(t)\tilde{\mathbf{z}}(t), \quad t \in \mathbb{N} \tag{12}$$

where $\tilde{\mathbf{z}}(t)=(\tilde{\mathbf{z}}_1^{\mathsf{T}}(t)\tilde{\mathbf{z}}_2^{\mathsf{T}}(t)\tilde{\mathbf{z}}_3^{\mathsf{T}}(t))^{\mathsf{T}}$, $\tilde{\mathbf{z}}_1(t) \in \mathbb{R}^n$, $\tilde{\mathbf{z}}_2(t) \in \mathbb{R}^n$, $\tilde{\mathbf{z}}_3(t) \in \mathbb{R}^n$, and

$$\tilde{\mathbf{A}}(t)=\begin{cases}\mathbf{\Lambda}_0, & t \in \{t_k+1,\cdots,t_{k+1}-q_k\}\\ & k \in \mathbb{N}\\ \mathbf{\Lambda}_1, & t \in \{t_{k+1}-q_k+1,\cdots,t_{k+1}-1\}\\ & k \in \mathbb{N}\\ \mathbf{\Lambda}_2, & t=t_{k+1}, k \in \mathbb{N}\end{cases} \tag{13}$$

$$\mathbf{\Lambda}_0 = \begin{pmatrix} \mathbf{A} & \mathbf{BK} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{A}}+\mathbf{BK} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{A}}+\mathbf{BK} & \mathbf{0} \end{pmatrix} \tag{14}$$

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \mathbf{A} & \mathbf{BK} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{A}}+\mathbf{BK} & \mathbf{0} \\ -\mathbf{LC} & \mathbf{BK} & \hat{\mathbf{A}}+\mathbf{LC} \end{pmatrix} \tag{15}$$

$$\mathbf{\Lambda}_2 = \begin{pmatrix} \mathbf{A} & \mathbf{BK} & \mathbf{0} \\ -\mathbf{LC} & \mathbf{BK} & \hat{\mathbf{A}}+\mathbf{LC} \\ -\mathbf{LC} & \mathbf{BK} & \hat{\mathbf{A}}+\mathbf{LC} \end{pmatrix}. \tag{16}$$

Remark 1: Although (12) appears as a system of nonminimal order when $t \in \{t_k+1, \cdots, t_{k+1}-q_k\}$, the formulation (12) with a constant-size $\tilde{\mathbf{A}}(t)$ offers the description simplicity and the ease of mathematical reasoning. Of course, in the simulation of $\hat{G}_K$, we can utilize a formulation of minimal order to reduce computational cost.

From (1), (4) to (6), and (11) to (16), we can obtain the relationship between $\hat{P}_K$ and $\hat{G}_K$ under the condition that they both are driven by the same sequence $[r_k] \in \mathcal{W}_{NM}$. Given $\mathbf{x}(0) = \tilde{\mathbf{z}}_1(0)$ and $\mathbf{x}_e(0) = \tilde{\mathbf{z}}_2(0) = \tilde{\mathbf{z}}_3(0)$, we have $\mathbf{x}(t) = \tilde{\mathbf{z}}_1(t)$ and $\mathbf{x}_e(t) = \tilde{\mathbf{z}}_2(t)$, $\forall t \in \mathbb{N}$. Specifically, we have $\mathbf{x}(t_k+1) = \tilde{\mathbf{z}}_1(t_k+1)$ and $\mathbf{x}_e(t_k+1) = \tilde{\mathbf{z}}_2(t_k+1) = \tilde{\mathbf{z}}_3(t_k+1)$.

Proposition 1: $\hat{G}_K$ is ES if and only if $\hat{P}_K$ is ES.

Proof: (If). $\forall \tilde{\mathbf{z}}(0) \in \mathbb{R}^{3n}$ and $\forall [r_k] \in \mathcal{W}_{NM}$, we have $\tilde{\mathbf{z}}_2(t_1+1) = \tilde{\mathbf{z}}_3(t_1+1)$ due to the expression of $\mathbf{\Lambda}_2$ in (16). Set $\mathbf{x}(0) = \tilde{\mathbf{z}}_1(t_1+1)$ and $\mathbf{x}_e(0) = \tilde{\mathbf{z}}_2(t_1+1) = \tilde{\mathbf{z}}_3(t_1+1)$. Let the sequence that drives $\hat{P}_K$ be $[r_1, r_2, \cdots]$. Then $\forall k \in \mathbb{N}$, $\tilde{\mathbf{z}}_1(t_{k+1}+1) = \mathbf{x}(t_{k+1}-t_1)$ and $\tilde{\mathbf{z}}_2(t_{k+1}+1) = \tilde{\mathbf{z}}_3(t_{k+1}+1) = \mathbf{x}_e(t_{k+1}-t_1)$. Because $\hat{P}_K$ is ES, $\|\mathbf{z}(t)\|^2 < \beta_1 \alpha^t \|\mathbf{z}(0)\|^2$, $\forall t \in \mathbb{N}$, where $0 < \alpha < 1$ and $\beta_1 > 1$ are constants independent of $\tilde{\mathbf{z}}(0)$ and $[r_k]$. Since $h_k \le N$, $\exists \beta_2 > 1$ independent of $\tilde{\mathbf{z}}(0)$ and $[r_k]$ such that $\forall k \in \mathbb{N}$ and $\forall \tau_k \in \{1, \cdots, h_k\}$,

$$\|\tilde{\mathbf{z}}(t_k+1+\tau_k)\|^2 < \beta_2 \alpha^{\tau_k} \|\tilde{\mathbf{z}}(t_k+1)\|^2.$$

Consequently, $\|\tilde{\mathbf{z}}(\tau_0)\|^2 < \beta_2 \alpha^{\tau_0} \|\tilde{\mathbf{z}}(0)\|^2$ and

$$\begin{aligned} \|\tilde{\mathbf{z}}(t_{k+1}+1+\tau_{k+1})\|^2 &< \beta_2 \alpha^{\tau_{k+1}} \|\tilde{\mathbf{z}}(t_{k+1}+1)\|^2 \\ &\le 2\beta_2 \alpha^{\tau_{k+1}} \|\mathbf{z}(t_{k+1}-t_1)\|^2 \\ &< 2\beta_2\beta_1 \alpha^{\tau_{k+1}+t_{k+1}-t_1} \|\mathbf{z}(0)\|^2 \\ &\le 2\beta_2\beta_1 \alpha^{\tau_{k+1}+t_{k+1}-t_1} \|\tilde{\mathbf{z}}(t_1+1)\|^2 \\ &< 2\beta_2^2\beta_1 \alpha^{\tau_{k+1}+t_{k+1}+1} \|\tilde{\mathbf{z}}(0)\|^2 \end{aligned}$$

which indicate that $\forall t \in \mathbb{N}$, $\|\tilde{\mathbf{z}}(t)\|^2 < 2\beta_2^2\beta_1\alpha^t \|\tilde{\mathbf{z}}(0)\|^2$. Hence, $\hat{G}_K$ is ES.

(Only If). $\forall \mathbf{z}(0) \in \mathbb{R}^{2n}$ and $\forall [r_k] \in \mathcal{W}_{NM}$, set $\tilde{\mathbf{z}}_1(0) = \mathbf{x}(0)$, $\tilde{\mathbf{z}}_2(0) = \mathbf{x}_e(0)$ and $\tilde{\mathbf{z}}_3(0) = \mathbf{x}_e(0)$ as well as let $\hat{G}_K$ be driven by the same sequence $[r_k]$ of update quality as $\hat{P}_K$. Then $\mathbf{x}(t) = \tilde{\mathbf{z}}_1(t)$ and $\mathbf{x}_e(t) = \tilde{\mathbf{z}}_2(t)$, $\forall t \in \mathbb{N}$. Because $\hat{G}_K$ is ES, we have

$$\|\mathbf{z}(t)\|^2 \le \|\tilde{\mathbf{z}}(t)\|^2 < \beta\alpha^t \|\tilde{\mathbf{z}}(0)\|^2 \le 2\beta\alpha^t \|\mathbf{z}(0)\|^2, \ \forall t \in \mathbb{N},$$

where $0 < \alpha < 1$ and $\beta > 1$ are constants independent of $\mathbf{z}(0)$ and $[r_k]$. Thus, $\hat{P}_K$ is ES. ■

By a similar procedure, it is easy to prove the following proposition.

Proposition 2: Under Assumption 1, $\hat{G}_K$ is EMSS if and only if $\hat{P}_K$ is EMSS.

Let us consider the auxiliary system $\hat{G}_{Ks}$ of $\hat{G}_K$, which is constructed as

$$\overline{\mathbf{z}}(k+1) = \overline{\mathbf{A}}(k)\overline{\mathbf{z}}(k), \quad k \in \mathbb{N} \tag{17}$$

where $\overline{\mathbf{A}}(k) = \mathbf{\Lambda}_2\mathbf{\Lambda}_1^{q_k-1}\mathbf{\Lambda}_0^{h_k-q_k}$. From (12), it is easy to see the following relationships between $\hat{G}_{Ks}$ and $\hat{G}_K$. Given $t_0 = -1$ and $\overline{\mathbf{z}}(0) = \tilde{\mathbf{z}}(t_0+1)$, then

$$\overline{\mathbf{z}}(k) = \tilde{\mathbf{z}}(t_k+1), \quad \forall k \in \mathbb{N}. \tag{18}$$

This result implies that if $\|\tilde{\mathbf{z}}(t)\|^2 < \beta\alpha^t \|\tilde{\mathbf{z}}(0)\|^2$, $0 < \alpha < 1$ and $\beta > 1$, then

$$\begin{aligned} \|\overline{\mathbf{z}}(k)\|^2 &< \beta\alpha^{t_k+1} \|\tilde{\mathbf{z}}(0)\|^2 = \beta \prod_{l=0}^{k-1} \alpha^{h_l} \|\tilde{\mathbf{z}}(0)\|^2 \\ &\le \beta\alpha^k \|\overline{\mathbf{z}}(0)\|^2. \end{aligned}$$

On the other hand, as h_k is bounded by N, $\exists \eta > 1$ independent of k and δ_k such that $\forall k \in \mathbb{N}$, $\|\tilde{\mathbf{z}}(t_k+1+\delta_k)\|^2 < \eta\|\tilde{\mathbf{z}}(t_k+1)\|^2$ for any $\delta_k \in \{0, \cdots, h_k-1\}$. The relationship (18) implies that if $\|\overline{\mathbf{z}}(k)\|^2 < \beta\alpha^k\|\overline{\mathbf{z}}(0)\|^2$ then for $t \in \{t_k+1, \cdots, t_{k+1}\}$ with $t < N(k+1)$, we have $\|\tilde{\mathbf{z}}(t)\|^2 < \eta\|\tilde{\mathbf{z}}(t_k+1)\|^2 < \eta\beta\alpha^k\|\overline{\mathbf{z}}(0)\|^2 < \eta(\beta/\alpha)(\alpha^{1/N})^t\|\tilde{\mathbf{z}}(0)\|^2$. Thus, we have the following.

Proposition 3: $\hat{G}_K$ is ES if and only if $\hat{G}_{Ks}$ is ES.

By a similar procedure, we can also derive the following proposition.

Proposition 4: Under Assumption 1, $\hat{G}_K$ is EMSS if and only if $\hat{G}_{Ks}$ is EMSS.

We are now ready to study the stability conditions for $\hat{P}_K$ with $\mathbf{A} \in \mathcal{A}$. For $h \in \mathcal{N}$ and $\mathcal{A}$ given in (10), first define

$$\mathcal{S}(h) \triangleq \left\{ \mathbf{S} = \prod_{i=1}^{h} \mathbf{A}_{v_i} | i \in \{1, 2, \cdots, h\}, v_i \in \{1, 2, \cdots, l\} \right\} \tag{19}$$

and then $\forall d \in \{0, 1, \cdots, h\}$, define a truncation mapping $\mathbf{\Gamma}_d$ from $\mathcal{S}(h)$ to $\mathbb{R}^{n\times n}$

$$\mathbf{\Gamma}_d(\mathbf{S}) \triangleq \mathbf{\Gamma}_d\left(\prod_{i=1}^{h} \mathbf{A}_{v_i}\right) = \begin{cases} \prod_{i=1}^{d} \mathbf{A}_{v_i}, & h \ge d > 0 \\ \mathbf{I}, & d = 0. \end{cases} \tag{20}$$

Furthermore, for $\mathbf{A} = \sum_{j=1}^{l} \omega_j \mathbf{A}_j \in \mathcal{A}$, a mapping ρ_A from $\mathcal{S}(h)$ to $\mathbb{R}$ can be established as

$$\rho_A(\mathbf{S}) = \rho_A\left(\prod_{i=1}^{h} \mathbf{A}_{v_i}\right) = \prod_{i=1}^{h} \omega_{v_i}. \tag{21}$$

For example, consider the case where $h = 3$ and $\mathcal{A}$ is determined by $\mathbf{A}_1$ and $\mathbf{A}_2$. The matrix set (19) is given by

$$\mathcal{S}(h) = \{\mathbf{A}_1^3, \mathbf{A}_1^2\mathbf{A}_2, \mathbf{A}_1\mathbf{A}_2\mathbf{A}_1, \mathbf{A}_1\mathbf{A}_2^2, \mathbf{A}_2\mathbf{A}_1^2, \mathbf{A}_2\mathbf{A}_1\mathbf{A}_2, \mathbf{A}_2^2\mathbf{A}_1, \mathbf{A}_2^3\}.$$

For the mapping (20) of $d = 2$, we have for instance

$$\begin{aligned} \mathbf{\Gamma}_2(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_1) &= \mathbf{\Gamma}_2(\mathbf{A}_1\mathbf{A}_2^2) = \mathbf{A}_1\mathbf{A}_2 \\ \mathbf{\Gamma}_2(\mathbf{A}_2^2\mathbf{A}_1) &= \mathbf{\Gamma}_2(\mathbf{A}_2^3) = \mathbf{A}_2^2. \end{aligned}$$

Given $\mathbf{A} = 0.3\mathbf{A}_1 + 0.7\mathbf{A}_2 \in \mathcal{A}$, we can check that the mapping (21) satisfies $\rho_A(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_1) = 0.3\times0.7\times0.3 = 0.063$ and $\rho_A(\mathbf{A}_2^2\mathbf{A}_1) = 0.7^2 \times 0.3 = 0.147$.

By the definition of the inverse mapping f^{-1} given in (8), $\overline{\mathbf{A}}(k)$ in (17) can be expressed as

$$\overline{\mathbf{A}}(k) = \mathbf{\Lambda}_2\mathbf{\Lambda}_1^{g_q(r_k)-1}\mathbf{\Lambda}_0^{g_h(r_k)-g_q(r_k)} \triangleq \mathbf{\Phi}(\mathbf{A}, r_k).$$

This together with (14) to (16) yields[1]

$$\boldsymbol{\Phi}(\mathbf{A},r)=\begin{pmatrix} \mathbf{A}^h & \sum_{i=0}^{h-1}\mathbf{A}^i\boldsymbol{\Theta}_1(h-1-i) & \mathbf{0} \\ \sum_{i=h-q}^{h-1}\boldsymbol{\Theta}_2(h-1-i)\mathbf{A}^i & \boldsymbol{\Phi}_1(r) & \boldsymbol{\Phi}_2(r) \\ \sum_{i=h-q}^{h-1}\boldsymbol{\Theta}_2(h-1-i)\mathbf{A}^i & \boldsymbol{\Phi}_1(r) & \boldsymbol{\Phi}_2(r)\end{pmatrix} \tag{22}$$

for $r \in \mathcal{R}_{NM}$, where $h = g_h(r)$, $q = g_q(r)$, $\boldsymbol{\Theta}_1(i) = \mathbf{BK}(\hat{\mathbf{A}} + \mathbf{BK})^i$, $\boldsymbol{\Theta}_2(i) = -(\hat{\mathbf{A}} + \mathbf{LC})^i\mathbf{LC}$

$$\boldsymbol{\Phi}_2(r) = \begin{cases}\mathbf{0}, & h > q\\ (\hat{\mathbf{A}}+\mathbf{LC})^h, & h = q,\end{cases}$$

$$\boldsymbol{\Phi}_1(r) = \begin{cases}\sum_{i=0}^{h-2}\sum_{j=0}^{\min(q-1,h-2-i)}\boldsymbol{\Theta}_2(j)\mathbf{A}^i\boldsymbol{\Theta}_1(h-2-i-j) \\ \qquad +\boldsymbol{\Theta}_3(h,q), \quad h>q \\ \sum_{i=0}^{h-2}\sum_{j=0}^{h-2-i}\boldsymbol{\Theta}_2(j)\mathbf{A}^i\boldsymbol{\Theta}_1(h-2-i-j)+\boldsymbol{\Theta}_4(h), \\ \qquad h=q\end{cases}$$

$$\begin{aligned}\boldsymbol{\Theta}_3(h,q) &= \sum_{i=0}^{q-1}(\hat{\mathbf{A}}+\mathbf{LC})^i\mathbf{BK}(\hat{\mathbf{A}}+\mathbf{BK})^{h-1-i} \\ &\quad + (\hat{\mathbf{A}}+\mathbf{LC})^q(\hat{\mathbf{A}}+\mathbf{BK})^{h-q} \\ \boldsymbol{\Theta}_4(h) &= \sum_{i=0}^{h-1}(\hat{\mathbf{A}}+\mathbf{LC})^i\mathbf{BK}(\hat{\mathbf{A}}+\mathbf{BK})^{h-1-i}.\end{aligned}$$

Furthermore, $\forall r \in \mathcal{R}_{NM}$ and $\forall \mathbf{S} \in \mathcal{S}(g_h(r))$, we denote

$$\boldsymbol{\Psi}(\mathbf{S},r) = \begin{pmatrix} \mathbf{S} & \sum_{i=0}^{h-1}\boldsymbol{\Gamma}_i(\mathbf{S})\boldsymbol{\Theta}_1(h-1-i) & \mathbf{0} \\ \sum_{i=h-q}^{h-1}\boldsymbol{\Theta}_2(h-1-i)\boldsymbol{\Gamma}_i(\mathbf{S}) & \boldsymbol{\Psi}_1(\mathbf{S},r) & \boldsymbol{\Phi}_2(r) \\ \sum_{i=h-q}^{h-1}\boldsymbol{\Theta}_2(h-1-i)\boldsymbol{\Gamma}_i(\mathbf{S}) & \boldsymbol{\Psi}_1(\mathbf{S},r) & \boldsymbol{\Phi}_2(r)\end{pmatrix} \tag{23}$$

with $h = g_h(r)$, $q = g_q(r)$, and

$$\boldsymbol{\Psi}_1(\mathbf{S},r)=\begin{cases}\sum_{i=0}^{h-2}\sum_{j=0}^{\min(q-1,h-2-i)}\boldsymbol{\Theta}_2(j)\boldsymbol{\Gamma}_i(\mathbf{S})\boldsymbol{\Theta}_1(h-2-i-j) \\ \quad +\boldsymbol{\Theta}_3(h,q),\ h>q \\ \sum_{i=0}^{h-2}\sum_{j=0}^{h-2-i}\boldsymbol{\Theta}_2(j)\boldsymbol{\Gamma}_i(\mathbf{S})\boldsymbol{\Theta}_1(h-2-i-j) \\ \quad +\boldsymbol{\Theta}_4(h),\ h=q.\end{cases}$$

Proposition 5: $\forall \mathbf{A} \in \mathcal{A}$ and $\forall r \in \mathcal{R}_{NM}$,

$$\boldsymbol{\Phi}(\mathbf{A},r) = \sum_{\mathbf{S}\in\mathcal{S}(g_h(r))}\rho_A(\mathbf{S})\boldsymbol{\Psi}(\mathbf{S},r) \tag{24}$$

with $\sum_{\mathbf{S}\in\mathcal{S}(g_h(r))}\rho_A(\mathbf{S}) = 1$, $\rho_A(\mathbf{S}) \geq 0$, $\forall \mathbf{S} \in \mathcal{S}(g_h(r))$.

Proof: It is easy to see that $\forall k \in \{1,\cdots,g_h(r)\}$,

$$\begin{aligned}\mathbf{A}^k &= \left(\sum_{j=1}^{l}\omega_j\mathbf{A}_j\right)^k\left(\sum_{j=1}^{l}\omega_j\mathbf{I}\right)^{g_h(r)-k} \\ &= \sum_{\mathbf{S}\in\mathcal{S}(g_h(r))}\rho_A(\mathbf{S})\boldsymbol{\Gamma}_k(\mathbf{S})\end{aligned} \tag{25}$$

$$\sum_{\mathbf{S}\in\mathcal{S}(g_h(r))}\rho_A(\mathbf{S}) = \left(\sum_{j=1}^{l}\omega_j\right)^{g_h(r)} = 1. \tag{26}$$

Then (24) follows from (22), (23), (25), and (26). Finally, $\rho_A(\mathbf{S}) = \prod_{i=1}^{g_h(r)}\omega_{v_i} \geq 0$ due to $\omega_j \geq 0$ for any $j \in \{1,\cdots,l\}$. ■

The proof of Proposition 5 can be illustrated with an simple example of $h = 2$ and $\mathbf{A} = 0.3\mathbf{A}_1 + 0.7\mathbf{A}_2 \in \mathcal{A}$. For this example, (25) and (26) become

$$\begin{aligned}\mathbf{A}^2 &= (0.3\mathbf{A}_1+0.7\mathbf{A}_2)^2 \\ &= 0.09\mathbf{A}_1^2+0.21\mathbf{A}_1\mathbf{A}_2+0.21\mathbf{A}_2\mathbf{A}_1+0.49\mathbf{A}_2^2 \\ &= \rho_A\left(\mathbf{A}_1^2\right)\mathbf{A}_1^2+\rho_A(\mathbf{A}_1\mathbf{A}_2)\mathbf{A}_1\mathbf{A}_2 \\ &\quad +\rho_A(\mathbf{A}_2\mathbf{A}_1)\mathbf{A}_2\mathbf{A}_1+\rho_A\left(\mathbf{A}_2^2\right)\mathbf{A}_2^2 \\ \mathbf{A} &= (0.3\mathbf{A}_1+0.7\mathbf{A}_2)(0.3+0.7) \\ &= 0.09\mathbf{A}_1+0.21\mathbf{A}_1+0.21\mathbf{A}_2+0.49\mathbf{A}_2 \\ &= \rho_A\left(\mathbf{A}_1^2\right)\boldsymbol{\Gamma}_1\left(\mathbf{A}_1^2\right)+\rho_A(\mathbf{A}_1\mathbf{A}_2)\boldsymbol{\Gamma}_1(\mathbf{A}_1\mathbf{A}_2) \\ &\quad +\rho_A(\mathbf{A}_2\mathbf{A}_1)\boldsymbol{\Gamma}_1(\mathbf{A}_2\mathbf{A}_1)+\rho_A\left(\mathbf{A}_2^2\right)\boldsymbol{\Gamma}_1\left(\mathbf{A}_2^2\right) \\ 1 &= (0.3+0.7)^2 = \rho_A\left(\mathbf{A}_1^2\right)+\rho_A(\mathbf{A}_1\mathbf{A}_2) \\ &\quad +\rho_A(\mathbf{A}_2\mathbf{A}_1)+\rho_A\left(\mathbf{A}_2^2\right).\end{aligned}$$

These three equations together with (22) and (23) lead to

$$\begin{aligned}\boldsymbol{\Phi}(\mathbf{A},r) &= \rho_A\left(\mathbf{A}_1^2\right)\boldsymbol{\Psi}\left(\mathbf{A}_1^2,r\right)+\rho_A(\mathbf{A}_1\mathbf{A}_2)\boldsymbol{\Psi}(\mathbf{A}_1\mathbf{A}_2,r) \\ &\quad +\rho_A(\mathbf{A}_2\mathbf{A}_1)\boldsymbol{\Psi}(\mathbf{A}_2\mathbf{A}_1,r)+\rho_A\left(\mathbf{A}_2^2\right)\boldsymbol{\Psi}\left(\mathbf{A}_2^2,r\right)\end{aligned}$$

which confirms the relationship (24).

Proposition 5 shows that for any $\mathbf{A} \in \mathcal{A}$, $\boldsymbol{\Phi}(\mathbf{A},r)$ is included in the convex polytope determined by $\boldsymbol{\Psi}(\mathbf{S},r)$ with $\mathbf{S} \in \mathcal{S}(g_h(r))$. Thus, we can treat an unknown $\boldsymbol{\Phi}(\mathbf{A},r)$ through treating the known $\boldsymbol{\Psi}(\mathbf{S},i)$, and this forms the basis for Theorem 1. First, the following lemma links $\boldsymbol{\Phi}(\mathbf{A},r)$ to exponential stability.

Lemma 1: (See [19]): $\hat{G}_{Ks}$ is ES if $\exists \mathbf{P} > \mathbf{0}$ such that $\boldsymbol{\Phi}^{\mathsf{T}}(\mathbf{A},i)\mathbf{P}\boldsymbol{\Phi}(\mathbf{A},i) - \mathbf{P} < \mathbf{0}, \forall i \in \mathcal{R}_{NM}$.

Theorem 1: $\hat{P}_K$ is ES for any $\mathbf{A} \in \mathcal{A}$ if $\exists \mathbf{P} > \mathbf{0}$ such that

$$\boldsymbol{\Psi}^{\mathsf{T}}(\mathbf{S},i)\mathbf{P}\boldsymbol{\Psi}(\mathbf{S},i) - \mathbf{P} < \mathbf{0}, \forall \mathbf{S} \in \mathcal{S}\left(g_h(i)\right), \forall i \in \mathcal{R}_{NM}. \tag{27}$$

Proof: According to Schur complement [9], (27) is equivalent to

$$\begin{pmatrix} -\mathbf{P} & \boldsymbol{\Psi}^{\mathsf{T}}(\mathbf{S},i)\mathbf{P} \\ \mathbf{P}\boldsymbol{\Psi}(\mathbf{S},i) & -\mathbf{P}\end{pmatrix} < \mathbf{0}. \tag{28}$$

Multiplying (28) by $\rho_A(\mathbf{S})$ and summing the resulting inequality over all $\mathbf{S} \in \mathcal{S}(g_h(i))$ lead to

$$\begin{pmatrix} -\sum_{\mathbf{S}}\rho_A(\mathbf{S})\mathbf{P} & \sum_{\mathbf{S}}\rho_A(\mathbf{S})\boldsymbol{\Psi}^{\mathsf{T}}(\mathbf{S},i)\mathbf{P} \\ \sum_{\mathbf{S}}\rho_A(\mathbf{S})\mathbf{P}\boldsymbol{\Psi}(\mathbf{S},i) & -\sum_{\mathbf{S}}\rho_A(\mathbf{S})\mathbf{P}\end{pmatrix} < \mathbf{0}.$$

This together with Proposition 5 establishes that, $\forall i \in \mathcal{R}_{NM}$

$$\begin{pmatrix} -\mathbf{P} & \boldsymbol{\Phi}^{\mathsf{T}}(\mathbf{A},i)\mathbf{P} \\ \mathbf{P}\boldsymbol{\Phi}(\mathbf{A},i) & -\mathbf{P}\end{pmatrix} < \mathbf{0}$$

is equivalent to

$$\boldsymbol{\Phi}^{\mathsf{T}}(\mathbf{A},i)\mathbf{P}\boldsymbol{\Phi}(\mathbf{A},i) - \mathbf{P} < \mathbf{0}.$$

[1]This note prescribes $\sum_{i=0}^{-1}\mathbf{X}_i = \mathbf{0}$.

According to Propositions 1 and 3 as well as Lemma 1, therefore, $\hat{P}_K$ is ES. ∎

Lemma 2: (see [14]) $\hat{G}_{Ks}$ under Assumption 1 is EMSS if $\exists\{\mathbf{P}(i) > \mathbf{0} : i \in \mathcal{R}_{NM}\}$ such that $\mathbf{\Phi}^{\mathsf{T}}(\mathbf{A}, i)\sum_{j=1}^{\bar{r}} p_{ij}\mathbf{P}(j)\mathbf{\Phi}(\mathbf{A}, i) - \mathbf{P}(i) < \mathbf{0}, \forall i \in \mathcal{R}_{NM}$.

Similar to the proof of Theorem 1, using Propositions 2, 4, and 5 as well as Lemma 2, we can derive Theorem 2.

Theorem 2: $\hat{P}_K$ under Assumption 1 is EMSS for any $\mathbf{A} \in \mathcal{A}$ if $\exists\{\mathbf{P}(i) > \mathbf{0} : i \in \mathcal{R}_{NM}\}$ such that

$$\mathbf{\Psi}^{\mathsf{T}}(\mathbf{S}, i)\sum_{j=1}^{\bar{r}} p_{ij}\mathbf{P}(j)\mathbf{\Psi}(\mathbf{S}, i) - \mathbf{P}(i) < \mathbf{0}, \quad \forall \mathbf{S} \in \mathcal{S}(g_h(i)),\ \forall i \in \mathcal{R}_{NM}. \tag{29}$$

Note that, for a given pair of $(\mathbf{K}, \mathbf{L})$, the linear matrix inequalities (LMIs) in Theorems 1 and 2 may be infeasible. If this situation arises, we can rearrange some conditions of the NCS, for example, decreasing N, increasing M or even redesigning $(\mathbf{K}, \mathbf{L})$, so that these LMIs are met.

IV. A Numerical Example

We considered the system where the matrices of the plant model were given by

$$\hat{\mathbf{A}} = \begin{pmatrix} 0.4 & 0.1 & 0.15 \\ -0.3 & 1.1 & 0 \\ 0.9 & 0.2 & -0.3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0.5 \\ -1 \\ 1 \end{pmatrix}$$
$$\mathbf{C} = (0.06 \quad -1 \quad 0.23).$$

The vertices of $\mathcal{A}$ were given by $\mathbf{A}_i = \hat{\mathbf{A}} + \rho\mathbf{\Delta}_i, i \in \{1, 2, 3\}$, with a scalar $0 \le \rho \in \mathbb{R}$ and

$$\mathbf{\Delta}_1 = \begin{pmatrix} -0.6 & -0.2 & 0.1 \\ -0.3 & 0.4 & 0.4 \\ 0.1 & -0.3 & -0.5 \end{pmatrix}$$
$$\mathbf{\Delta}_2 = \begin{pmatrix} 0.2 & 0.1 & 0.5 \\ 0.2 & -0.6 & 0.1 \\ 0.4 & 0.3 & 0.2 \end{pmatrix}$$
$$\mathbf{\Delta}_3 = \begin{pmatrix} 0.2 & 0.4 & -0.3 \\ 0.3 & 0.2 & 0.1 \\ -0.5 & 0.1 & 0.6 \end{pmatrix}.$$

The state feedback gain and estimator gain matrices were given, respectively, by

$$\mathbf{K} = (-0.43 \quad 0.35 \quad 0.17)$$
$$\mathbf{L} = (-0.47 \quad 0.75 \quad 0.10)^{\mathsf{T}}.$$

The values of N and M were set to $N = 3$ and $M = 2$, which yielded $\bar{r} = 5$. The aim was to make ρ as large as possible without the loss of exponential stability of $\hat{P}_K$ for any $\mathbf{A} \in \mathcal{A}$.

For a given value of ρ, the sufficient stability condition (27) in Theorem 1 actually consists of $3^3 + 3^3 + 3^2 + 3^2 + 3^1 = 75$ LMIs. Hence, with a combination of LMI technique and bisection search, we convinced that $\hat{P}_K$ maintains the exponential stability for any $\mathbf{A} \in \mathcal{A}$ when $\rho \le \rho_1 = 0.3124$. We furthermore considered this $\hat{P}_K$ under Assumption 1 with

$$\mathbf{\Upsilon} = \begin{pmatrix} 0.5 & 0.2 & 0.2 & 0.1 & 0 \\ 0.2 & 0.5 & 0.2 & 0.1 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 & 0 \\ 0 & 0.1 & 0.2 & 0.5 & 0.2 \\ 0 & 0.1 & 0.2 & 0.2 & 0.5 \end{pmatrix}.$$

By Theorem 2 and bisection search, we convinced that $\hat{P}_K$ under Assumption 1 maintains the exponential mean square stability for any $\mathbf{A} \in \mathcal{A}$ when $\rho \le \rho_2 = 0.3352$. The observation of $\rho_2 > \rho_1$ is in accordance with the fact that (27) is a sufficient condition of (29).

V. Conclusions

Stability properties have been analyzed for discrete-time NCSs with polytopic uncertainty under buffer capacity constraint, where a controller is updated with the buffered sensor data at random intervals and the amount of the buffered data received by the controller is also random. We have established the sufficient conditions for guaranteeing the exponential stability of generic switched NCSs and the exponential mean square stability of Markov-chain driven NCSs, respectively. An example has been given to illustrate our method.

The discrete-time plant can in fact be obtained by a discretization of the corresponding continuous-time plant, with a zero-holder and a sampler of the fixed sampling period. The works [5], [6], [8], [10], [12], and [20] also employ discrete-time approaches for continuous-time NCSs or continuous-time closed-loop systems with delayed control. Although the state feedback employed in these works is simpler than our observer-based state feedback, these studies consider varying communication delay, a network separating the controller and actuator, and/or varying sampling period, which are more practical and complicated than our assumptions of the "collocated" controller and actuator and the update interval being a multiple of the sampling period. Extension of our results to more general NCSs is currently under investigation. In addition, since a full-order observer-based controller is utilised, our method is applicable to the single link case. How to develop an ingenious structure of observer-based controller accommodating multiple links is an interesting open problem.

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Joint Estimation and Gossip Averaging for Sensor Network Applications

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***Abstract*—This note presents an efficient distributed approach for computing a spatial average of parameters estimated by sensors in a wireless network. The most intuitive approach would rely on a two-step procedure. First, the nodes would estimate the local quantities, and second a distributed process would average these estimates over the network. Instead, the proposed algorithm combines both processes to foster the convergence while fulfilling the usual wireless sensor network requirements: simplicity, low memory/CPU usage, and asynchronicity.**

***Index Terms*—Average consensus, averaging, distributed algorithms, epidemic algorithms, estimation, gossip algorithms, sensor networks, space-time diffusion.**

I. Introduction

Wireless sensor networks (WSN) consist of a great amount of small entities, called nodes, equipped with low cost hardware in order to contain the total network cost. They are commonly used for tracking purposes and environmental monitoring, such as landslides or fires detection [1]. The direct drawbacks of using low cost hardware are numerous: severe energy constraints (battery lifetime), poor CPU and storage abilities, low transmission rates and small communication ranges. Due to these constraints, distributed algorithms often outperform centralized approaches. However, the most important challenge for designing a distributed algorithm relies on ensuring a good performance (convergence, speed, accuracy...) at a low energetic cost. Many problems in WSN, such as synchronization or average estimation, rely on reaching a consensus over the network. Further, many other complex problems may be described as a consensus problem [2]. For these problems, the particular class of distributed consensus algorithms is of great interest: they provide a robust way for homogenizing parameters among network nodes [3]. More specifically, average consensus algorithms are required when the stability and the quality of the consensus point is a critical issue, and extend to a wide panel of data fusion tools such as estimators for statistical moments, linear regression and polynomial map fitting ([4], [5]). However, a restriction occurs when data to be averaged are subject to time fluctuations. This is the case, for instance, when a statistical parameter is estimated from time series of noisy data samples. The local estimation is obtained thanks to a temporal regularization process, performed independently by each sensor from its own measures. For an additive zero-mean stationary noise process, the accuracy and the stability of this estimation increase with the number of measures and, as a corollary, with time. When a given accuracy is achieved, the second process can start performing a spatial regularization over the network. This approach turns out to be inefficient in terms of energy and processing time, and further, the perfect convergence cannot be asymptotically achieved. Therefore, it seems more challenging to regularize with respect to time and space simultaneously. For this purpose, we propose to run the gossip averaging algorithm while the estimation is still in progress. We further introduce correction mechanisms to ensure an accurate asymptotic convergence. This double process is clearly a distributed space-time regularization scheme: each node performs individually a local regularization of measured data series, while a spatial regularization (averaging) is performed in order to extract a global characteristic. This issue was firstly addressed in [5]. In this note, we put forward a new algorithm which covers a more wide range of estimators and allows a full asynchrony in/between packet exchanges and estimation processes. This feature better complies with the constraints of WSNs. The proposed algorithm turns out to be applicable in many contexts: as an example, a clock carrier synchronization application for WSN is described in [6]. This note is organized as follows: Section II provides a short overview of gossip-based consensus algorithms, their principles and some known results. Section III describes the newly proposed algorithm, named the Joint Estimation and Gossip Averaging Algorithm (JEGA), which describes an asynchronous distributed algorithm for averaging a parameter over the network simultaneously with its local estimation. The convergence proof is provided. After these theoretical considerations, simulation results are provided in Section IV to illustrate the behavior of JEGA.

II. Distributed Consensus Algorithms

Distributed consensus algorithms/protocols aim at agreeing all network nodes with a common value or a decision in a decentralized fashion. From a signal processing context, this can be understood as a spatial regularization process. In [7], this class of algorithms is used to extract a large variety of aggregated and statistical quantities like averages/variances, max/min values\ldots When data exchanges consist of local, asynchronous and simple interactions between neighbor nodes, such algorithms referred to as gossip-based. The particular subclass of gossip-based average consensus algorithms ([7], [8] ...) aims at computing a global average of local values and has practical applications such as LMS-based fitting of a linear parametric model ([2], [4], [5]) or carrier frequency homogenization ([6]). These conceptual and applicative characteristics make gossip-based consensus algorithms

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