



Brief paper

Robust Finite Word Length controller design[☆]Jun Wu^a, Gang Li^b, Sheng Chen^{c,*}, Jian Chu^a^a State Key Lab of Industrial Control Technology, Institute of Cyber-Systems and Control, Zhejiang University, Hangzhou 310027, China^b College of Information Engineering, Zhejiang University of Technology, Hangzhou 310014, China^c School of Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, UK

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ABSTRACT

A novel Finite Word Length (FWL) controller design is proposed in the framework of a mixed μ theory. A robust FWL controller performance measure is first developed, which takes into account the standard robust control requirements as well as the FWL implementation considerations, and the corresponding controller design problem is naturally reformulated as a mixed μ problem which can be treated effectively with the results of the mixed μ theory.

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1. Introduction

Robust control capable of coping with uncertainty in plant dynamics has been the focal point of the control community for the past three decades. An implicit assumption in most of the existing robust design methods is that controllers are implemented exactly, i.e. there is no uncertainty occurring in realising controllers. In reality controllers are implemented with Finite Word Length (FWL) processors. In 1997, the fragility problem was raised in the work of (Keel & Bhattacharyya, 1997) which showed by examples that a controller achieving the largest robustness to plant uncertainty most likely has a vanishingly small closed-loop stability margin with respect to the controller parameters. Thus, a control system designed by maximising its robustness to plant uncertainty may be fragile, and the resulting fragile controller will need a processor with a very long bit length in implementation to minimise the FWL effects and therefore avoid degrading the designed closed-loop performance or even destabilising the designed stable closed-loop system. However, in many practical systems, such as

mass-produced electronic consumer goods, fixed-point processors of short word length are preferred because of their advantages in component cost, chip area, operation simplicity and power consumption. Therefore, it is not a practical approach to simply pursue the optimal robustness to plant uncertainty without considering the FWL effects (Franklin, Powell & Workman, 1998; Gevers & Li, 1993; Istepanian & Whidborne, 2001).

A suitable robust design approach is maintaining a suboptimal robustness to plant uncertainty while simultaneously making the controller tolerance to FWL implementation as large as possible. Through this design, a robust controller can be obtained which does not require a long word-length hardware for implementation. There exist two types of main FWL errors in digital controller implementation. The first one is the rounding errors that occur in arithmetic operations, and the second one is the parameter representation errors. Typically, these two types of errors are investigated separately for the reason of mathematical tractability. In this paper we deal with the second type of FWL errors. Specifically we consider FWL parameter representation errors in the design of robust controllers.

Most of the existing researches (Collins & Zhao, 2001; D'Andrea & Istepanian, 2002; Mahmoud, 2004, 2005; Norlander & Mäkilä, 2001; Park, 2004; Yang, Wang, & Soh, 2000, 2001; Yee, Yang & Wang, 2000; Yee, Yang, & Wang, 2001) refer to robust digital control design with the consideration of FWL parameter representation errors as non-fragile/defragile/resilient control. The works (Mahmoud, 2004, 2005; Park, 2004; Yang et al., 2000;

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Yee et al., 2001) hypothesize that the controller parameter perturbation block is 2-norm bounded, while the works (Collins & Zhao, 2001; D’Andrea & Istepanian, 2002; Norlander & Mäkilä, 2001; Yang et al., 2001; Yee et al., 2000) use a more suitable hypothesis which claims that every parameter perturbation is independent and is magnitude bounded. The methods of D’Andrea and Istepanian (2002) and Yee et al. (2000) deal with static state feedback, while the LQ control of a fixed order and PID control are studied in Norlander and Mäkilä (2001) and Collins and Zhao (2001), respectively. A common Lyapunov function matrix is sought in Yang et al. (2001) for all the vertices of the FWL perturbation hypercube in designing a H₂ controller of fixed order, but the huge number of these vertices can result in excessive complexity in practical computation.

Under the suitable hypothesis similar to the one used in Collins and Zhao (2001), D’Andrea and Istepanian (2002), Norlander and Mäkilä (2001), Yang et al. (2001) and Yee et al. (2000) we study the design of the H_∞ output feedback controller of a fixed order with FWL considerations. A novel FWL robust control performance measure is proposed which takes into account the standard robust control requirements, such as plant uncertainties and input–output characteristics, as well as the FWL effects on controller implementation. We show that the related robust FWL controller design problem can naturally be formulated as a mixed μ problem, and thus it can be solved effectively with the aid of the mixed μ theory. Our proposed robust FWL controller design is also computationally more attractive than the existing design methods, such as the one introduced in Yang et al. (2001) which suffers from high-dimensionality difficulty.

The remainder of this paper is organised in the following way. Notations and preliminaries are offered in Section 2. Section 3 presents a robust FWL performance measure, while Section 4 derives the proposed design approach through optimising this measure. Two numerical examples are given in Section 5 to demonstrate the effectiveness of our proposed method, and the paper concludes at Section 6.

2. Notations and preliminaries

Let \mathcal{R} be the field of real numbers and \mathcal{C} the field of complex numbers, while \mathcal{U} is the closed unit disk in \mathcal{C} . For a matrix \mathbf{A} , $\mathbf{A} > 0$ means that \mathbf{A} is a positive definite matrix, \mathbf{A}^T denotes the transpose of \mathbf{A} , and \mathbf{A}^* the complex conjugate transpose of \mathbf{A} . The largest singular value of \mathbf{A} is denoted by $\bar{\sigma}(\mathbf{A})$. $\|\mathbf{A}\|_F$ is the Frobenius norm of \mathbf{A} , while $\|\mathbf{A}\|_m$ is the modulus of the entry whose modulus is the largest among all the entries of \mathbf{A} . $\rho(\mathbf{A})$ and $\det \mathbf{A}$ represent the spectral radius and the determinant of square matrix \mathbf{A} , respectively. \mathbf{I}_n is the $n \times n$ identity matrix, while \mathbf{I} and $\mathbf{0}$ represent the identity and zero matrices of appropriate dimensions, respectively. Let $\mathbf{d}_n = [1 \ \dots \ 1] \in \mathcal{R}^{1 \times n}$ be the $1 \times n$ row vector whose elements are all equal to 1. $\mathbf{A} \otimes \mathbf{B}$ is the Kronecker product of matrices \mathbf{A} and \mathbf{B} .

Denote \mathcal{F} the set of all the causal finite-dimensional linear time-invariant discrete-time systems. Any system in \mathcal{F} can be described as

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k), \end{cases} \quad (1)$$

where the real constant matrices \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} have appropriate dimensions. The transfer function matrix of the above system is

$$\hat{\mathbf{G}}(w) \triangleq w\mathbf{C}(\mathbf{I} - w\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \quad (2)$$

$\hat{\mathbf{G}}(w)$ is stable (\mathbf{A} is stable) if and only if $\rho(\mathbf{A}) < 1$ or equivalently $\forall w \in \mathcal{U}, \det(\mathbf{I} - w\mathbf{A}) \neq 0$. The H_∞ norm and H₂ norm of stable $\hat{\mathbf{G}}(w)$ are defined as

$$\|\hat{\mathbf{G}}(w)\|_\infty \triangleq \sup_{w \in \mathcal{U}} \bar{\sigma}(\hat{\mathbf{G}}(w)) < \infty, \quad (3)$$

$$\|\hat{\mathbf{G}}(w)\|_2 \triangleq \left(\|\mathbf{D}\|_F^2 + \sum_{i=0}^{\infty} \|\mathbf{C}\mathbf{A}^i\mathbf{B}\|_F^2 \right)^{1/2} < \infty, \quad (4)$$

respectively. For a discrete-time stable system, its H_∞ norm is no less than its H₂ norm.

The following results of the mixed μ theory are from Young (1993). Suppose that we have a matrix $\mathbf{M} \in \mathcal{C}^{n_a \times n_a}$ and three non-negative integers p, q and r with $p + q + r \leq n_a$, which specify the numbers of uncertainty blocks of three types: repeated complex scalars, repeated real scalars and full complex blocks. A $(p + q + r)$ -tuple of positive integers

$$\mathbf{k}(p, q, r) = [k_1 \ \dots \ k_p \ k_{p+1} \ \dots \ k_{p+q} \ m_1 \ \dots \ m_r]^T \quad (5)$$

specifies the dimensions of the perturbation blocks, and

$$\sum_{i=1}^{p+q} k_i + \sum_{j=1}^r m_j = n_a$$

in order that these dimensions are compatible with \mathbf{M} . The block structure $\mathbf{k}(p, q, r)$ determines the set of allowable perturbations, namely,

$$\mathcal{K} \triangleq \left\{ \mathbf{Y} \left| \begin{array}{l} \mathbf{Y} = \text{diag}(\zeta_1 \mathbf{I}_{k_1}, \dots, \zeta_p \mathbf{I}_{k_p}, \\ \zeta_{p+1} \mathbf{I}_{k_{p+1}}, \dots, \zeta_{p+q} \mathbf{I}_{k_{p+q}}, \mathbf{\Gamma}_1, \dots, \mathbf{\Gamma}_r) \\ \forall i \in \{1, \dots, p\}, \zeta_i \in \mathcal{C} \\ \forall i \in \{p+1, \dots, p+q\}, \zeta_i \in \mathcal{R} \\ \forall j \in \{1, \dots, r\}, \mathbf{\Gamma}_j \in \mathcal{C}^{m_j \times m_j} \end{array} \right. \right\}.$$

The mixed μ of a matrix $\mathbf{M} \in \mathcal{C}^{n_a \times n_a}$ with respect to a perturbation set \mathcal{K} is defined as

$$\mu_{\mathcal{K}}(\mathbf{M}) \triangleq \left(\inf_{\mathbf{Y} \in \mathcal{K}} \{ \bar{\sigma}(\mathbf{Y}) \mid \det(\mathbf{I} - \mathbf{Y}\mathbf{M}) = 0 \} \right)^{-1}. \quad (6)$$

Lemma 1. Suppose that $p = 1, q = 0$ and $r = 0$. Then $\mu_{\mathcal{K}}(\mathbf{M}) = \rho(\mathbf{M})$.

Presently, except for a few special cases, how to compute $\mu_{\mathcal{K}}(\mathbf{M})$ is unknown. However, an upper bound of $\mu_{\mathcal{K}}(\mathbf{M})$ provided in the following is easy to compute and is often used to replace $\mu_{\mathcal{K}}(\mathbf{M})$ in practice. Define

$$\begin{aligned} \mathcal{E}_{\mathcal{K}} &\triangleq \left\{ \mathbf{E} \left| \begin{array}{l} \mathbf{E} = \text{diag}(\mathbf{E}_1, \dots, \mathbf{E}_{p+q}, \eta_1 \mathbf{I}_{m_1}, \dots, \eta_r \mathbf{I}_{m_r}) \\ \forall i \in \{1, \dots, p+q\}, 0 < \mathbf{E}_i \in \mathcal{C}^{k_i \times k_i} \\ \forall j \in \{1, \dots, r\}, 0 < \eta_j \in \mathcal{R} \end{array} \right. \right\} \\ \mathcal{G}_{\mathcal{K}} &\triangleq \left\{ \mathbf{G} \left| \begin{array}{l} \mathbf{G} = \text{diag}(\mathbf{0}\mathbf{I}_{k_1}, \dots, \mathbf{0}\mathbf{I}_{k_p}, \\ \mathbf{G}_{p+1}, \dots, \mathbf{G}_{p+q}, \mathbf{0}\mathbf{I}_{m_1}, \dots, \mathbf{0}\mathbf{I}_{m_r}) \\ \forall i \in \{p+1, \dots, p+q\}, \mathbf{G}_i = \mathbf{G}_i^* \in \mathcal{C}^{k_i \times k_i} \end{array} \right. \right\}. \end{aligned}$$

Then an upper bound of $\mu_{\mathcal{K}}(\mathbf{M})$ is

$$\alpha_{\mathcal{K}}(\mathbf{M}) \triangleq \inf_{\substack{\mathbf{E} \in \mathcal{E}_{\mathcal{K}} \\ \mathbf{G} \in \mathcal{G}_{\mathcal{K}} \\ 0 < \alpha \in \mathcal{R}}} \left\{ \alpha \left| \begin{array}{l} \alpha^2 \mathbf{E} - \mathbf{M}^* \mathbf{E} \mathbf{M} \\ -\sqrt{-1}(\mathbf{G}\mathbf{M} - \mathbf{M}^* \mathbf{G}) > 0 \end{array} \right. \right\}. \quad (7)$$

When the real scalars of $\mathbf{Y} \in \mathcal{K}$ are not repeated and \mathbf{M} is a real matrix, $\alpha_{\mathcal{K}}(\mathbf{M})$ can be expressed and computed more simply.

Define $\mathcal{E}_{\mathcal{R}, \mathcal{K}} \triangleq \{ \mathbf{E} \in \mathcal{E}_{\mathcal{K}} \mid \mathbf{E} \in \mathcal{R}^{n_a \times n_a} \}$.

Lemma 2. Suppose that we have a real matrix $\mathbf{M} \in \mathcal{R}^{n_a \times n_a}$ and a perturbation set \mathcal{K} with $k_i = 1$ for $i \in \{p+1, \dots, p+q\}$ (i.e. none of the real scalars are repeated). Then

$$\alpha_{\mathcal{K}}(\mathbf{M}) = \inf_{\substack{\mathbf{E} \in \mathcal{E}_{\mathcal{R}, \mathcal{K}} \\ 0 < \alpha \in \mathcal{R}}} \{\alpha \mid \alpha^2 \mathbf{E} - \mathbf{M}^T \mathbf{E} \mathbf{M} > 0\}. \quad (8)$$

Corollary 1. For \mathbf{M} and \mathcal{K} as in Lemma 2, $\alpha_{\mathcal{K}}(\mathbf{M}) < 1$ if and only if there exists $\mathbf{E} \in \mathcal{E}_{\mathcal{R}, \mathcal{K}}$ such that $\mathbf{E} - \mathbf{M}^T \mathbf{E} \mathbf{M} > 0$.

Consider a matrix $\mathbf{M} \in \mathcal{C}^{n_a \times n_a}$ partitioned as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{1,1} & \mathbf{M}_{1,2} \\ \mathbf{M}_{2,1} & \mathbf{M}_{2,2} \end{bmatrix}$$

with square $\mathbf{M}_{1,1}$ and $\mathbf{M}_{2,2}$. The perturbation sets \mathcal{K}_1 and \mathcal{K}_2 are compatible with $\mathbf{M}_{1,1}$ and $\mathbf{M}_{2,2}$, respectively. Then the perturbation set defined in (9) is compatible with \mathbf{M} .

$$\mathcal{K}_f \triangleq \{\mathbf{Y} = \text{diag}(\mathbf{Y}_1, \mathbf{Y}_2) \mid \mathbf{Y}_1 \in \mathcal{K}_1, \mathbf{Y}_2 \in \mathcal{K}_2\}. \quad (9)$$

Lemma 3. For $0 < \alpha \in \mathcal{R}$, $\mu_{\mathcal{K}_f}(\mathbf{M}) < \alpha$ if and only if $\mu_{\mathcal{K}_1}(\mathbf{M}_{1,1}) < \alpha$ and $\forall \mathbf{Y}_1 \in \mathcal{K}_1$ with $\bar{\sigma}(\mathbf{Y}_1) \leq \frac{1}{\alpha}$, $\mu_{\mathcal{K}_2}(F(\mathbf{M}, \mathbf{Y}_1)) < \alpha$, where

$$F(\mathbf{M}, \mathbf{Y}_1) \triangleq \mathbf{M}_{2,2} + \mathbf{M}_{2,1}(\mathbf{I} - \mathbf{Y}_1 \mathbf{M}_{1,1})^{-1} \mathbf{Y}_1 \mathbf{M}_{1,2}. \quad (10)$$

3. FWL robust performance measure \tilde{v}

The plant is described by a known nominal model $\hat{\mathbf{P}}_g(w)$ and an unknown but bounded structured uncertainty $\hat{\mathbf{U}}(w)$. The model $\hat{\mathbf{P}}_g(w)$ is given as

$$\begin{cases} \mathbf{x}_p(k+1) = \mathbf{A}_p \mathbf{x}_p(k) + \mathbf{B}_v \mathbf{v}(k) + \mathbf{B}_w \mathbf{w}(k) + \mathbf{B}_p \mathbf{u}_p(k), \\ \mathbf{h}(k) = \mathbf{C}_h \mathbf{x}_p(k) + \mathbf{D}_{1,1} \mathbf{v}(k) + \mathbf{D}_{1,2} \mathbf{w}(k), \\ \mathbf{z}(k) = \mathbf{C}_z \mathbf{x}_p(k) + \mathbf{D}_{2,1} \mathbf{v}(k) + \mathbf{D}_{2,2} \mathbf{w}(k) + \mathbf{D}_{2,3} \mathbf{u}_p(k), \\ \mathbf{y}_p(k) = \mathbf{C}_p \mathbf{x}_p(k) + \mathbf{D}_{3,2} \mathbf{w}(k), \end{cases} \quad (11)$$

where state $\mathbf{x}_p(k) \in \mathcal{R}^n$, uncertainty-linked input $\mathbf{v}(k) \in \mathcal{R}^{n_1}$, external disturbance input $\mathbf{w}(k) \in \mathcal{R}^{n_2}$, control input $\mathbf{u}_p(k) \in \mathcal{R}^s$, uncertainty-linked output $\mathbf{h}(k) \in \mathcal{R}^{n_1}$, controlled output $\mathbf{z}(k) \in \mathcal{R}^{n_2}$, and measured output $\mathbf{y}_p(k) \in \mathcal{R}^t$. Note that we have assumed without loss of generality that $\mathbf{v}(k)$ and $\mathbf{h}(k)$ have the same dimension as well as that $\mathbf{w}(k)$ and $\mathbf{z}(k)$ have the same dimension. If the dimensions of the paired two variables are different, they can always be made equal by adding an appropriate number of zero rows/columns to the corresponding plant matrices. In addition, it is assumed that $\mathbf{B}_p^T \mathbf{B}_p > 0$ and $\mathbf{C}_p \mathbf{C}_p^T > 0$. This assumption reflects a reasonable practical situation of no redundant actuator or sensor.

Through \mathbf{h} and \mathbf{v} , $\hat{\mathbf{P}}_g(w)$ connects with the structured uncertainty $\hat{\mathbf{U}}(w)$, i.e.

$$\mathbf{v} = \hat{\mathbf{U}}(w) \mathbf{h} = \text{diag}(\hat{\mathbf{U}}_1(w), \dots, \hat{\mathbf{U}}_{b+d}(w)) \mathbf{h}, \quad (12)$$

where $\hat{\mathbf{U}}_i(w) = \varphi_i(w) \mathbf{I}_{p_i}$ with $\varphi_i(w) \in \mathcal{C}$, $\forall w \in \mathcal{C}$, $\forall i \in \{1, \dots, b\}$, and $\hat{\mathbf{U}}_i(w) \in \mathcal{C}^{p_i \times p_i}$, $\forall w \in \mathcal{C}$, $\forall i \in \{b+1, \dots, b+d\}$, while

$$\sum_{i=1}^{b+d} p_i = n_1, \quad p_i \geq 1.$$

It is assumed that the above $\hat{\mathbf{U}}(w)$ is included in the set

$$\mathcal{H}_\tau \triangleq \left\{ \hat{\mathbf{U}}(w) \mid \begin{cases} \hat{\mathbf{U}}(w) = \text{diag}(\hat{\mathbf{U}}_1(w), \dots, \hat{\mathbf{U}}_{b+d}(w)) \\ \hat{\mathbf{U}}(w) \in \mathcal{F}, \hat{\mathbf{U}}(w) \text{ is stable,} \\ \|\hat{\mathbf{U}}(w)\|_\infty < \tau \end{cases} \right\}$$

with a given constant $\tau > 0$.

The digital controller $\hat{\mathbf{C}}(w)$ of the m th-order is described by

$$\begin{cases} \mathbf{x}_c(k+1) = \mathbf{A}_c \mathbf{x}_c(k) + \mathbf{B}_c \mathbf{y}_p(k) \\ \mathbf{u}_p(k) = \mathbf{C}_c \mathbf{x}_c(k) + \mathbf{D}_c \mathbf{y}_p(k) \end{cases} \quad (13)$$

with $\mathbf{A}_c \in \mathcal{R}^{m \times m}$, $\mathbf{B}_c \in \mathcal{R}^{m \times t}$, $\mathbf{C}_c \in \mathcal{R}^{s \times m}$ and $\mathbf{D}_c \in \mathcal{R}^{s \times t}$. Let us denote

$$\mathbf{X} \triangleq \begin{bmatrix} \mathbf{D}_c & \mathbf{C}_c \\ \mathbf{B}_c & \mathbf{A}_c \end{bmatrix} \in \mathcal{R}^{(s+m) \times (t+m)}.$$

When \mathbf{X} is implemented in a fixed-point format of FWL, it is perturbed into $\mathbf{X} + \mathbf{\Delta}$ with $\mathbf{\Delta}$ belonging to the hypercube

$$\mathcal{D}_\beta \triangleq \{\mathbf{\Delta} \mid \mathbf{\Delta} \in \mathcal{R}^{(s+m) \times (t+m)}, \|\mathbf{\Delta}\|_m \leq \beta\}, \quad (14)$$

where $0 \leq \beta \in \mathcal{R}$ is the maximum representation error of the fixed-point digital processor. Denote

$$\mathcal{N} \triangleq (s+m)(t+m), \quad (15)$$

$$\mathcal{O} \triangleq \{\mathbf{Q} \mid \mathbf{Q} \in \mathcal{R}^{N \times N}, \mathbf{Q} \text{ is diagonal}\}, \quad (16)$$

$$\mathcal{O}_\beta \triangleq \{\mathbf{Q} \mid \mathbf{Q} \in \mathcal{O}, \bar{\sigma}(\mathbf{Q}) \leq \beta\}. \quad (17)$$

Further express $\mathbf{\Delta}$ as

$$\mathbf{\Delta} \triangleq \begin{bmatrix} \delta_{1,1} & \delta_{1,2} & \cdots & \delta_{1,t+m} \\ \delta_{2,1} & \delta_{2,2} & \cdots & \delta_{2,t+m} \\ \vdots & \vdots & \cdots & \vdots \\ \delta_{s+m,1} & \delta_{s+m,2} & \cdots & \delta_{s+m,t+m} \end{bmatrix}. \quad (18)$$

It is easy to check that

$$\mathbf{X} + \mathbf{\Delta} = \mathbf{X} + (\mathbf{d}_{t+m} \otimes \mathbf{I}_{s+m}) \mathbf{\Lambda} (\mathbf{I}_{t+m} \otimes \mathbf{d}_{s+m}^T), \quad (19)$$

$$\mathbf{\Lambda} \triangleq \text{diag}(\delta_{1,1}, \delta_{2,1}, \dots, \delta_{s+m,1}, \delta_{1,2}, \dots, \delta_{s+m,2}, \dots, \delta_{1,t+m}, \dots, \delta_{s+m,t+m}) \in \mathcal{O}_\beta. \quad (20)$$

The above description represents a closed-loop system consisting of $\hat{\mathbf{P}}_g(w)$ and $\hat{\mathbf{U}}(w)$ as well as \mathbf{X} and $\mathbf{\Lambda}$. Denote this closed-loop system as $\hat{\Phi}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{\Lambda})$ and the closed-loop transfer function from $\mathbf{w}(k)$ to $\mathbf{z}(k)$ as $\hat{\Phi}_{wz}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{\Lambda})$. For $0 < \xi \in \mathcal{R}$, a set is defined which consists of all the m th-order robust controllers without FWL consideration, that is,

$$\mathcal{X}_m \triangleq \left\{ \mathbf{X} \mid \begin{cases} \mathbf{X} \in \mathcal{R}^{(s+m) \times (t+m)}, \forall \hat{\mathbf{U}}(w) \in \mathcal{H}_\tau, \\ \hat{\Phi}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{0}) \text{ is stable,} \\ \|\hat{\Phi}_{wz}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{0})\|_\infty \leq \xi \end{cases} \right\}. \quad (21)$$

To take into account the FWL error $\mathbf{\Lambda}$, we propose the following FWL performance measure for $\mathbf{X} \in \mathcal{X}_m$

$$v(\mathbf{X}) \triangleq \sup_{0 \leq \beta \in \mathcal{R}} \left\{ \beta \mid \begin{cases} \forall \hat{\mathbf{U}}(w) \in \mathcal{H}_\tau, \forall \mathbf{\Lambda} \in \mathcal{O}_\beta, \\ \hat{\Phi}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{\Lambda}) \text{ is stable,} \\ \|\hat{\Phi}_{wz}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{\Lambda})\|_\infty \leq \xi \end{cases} \right\}.$$

For a given $\mathbf{X} \in \mathcal{X}_m$, how to compute the value of $v(\mathbf{X})$ is unknown. Therefore, a tractable lower bound of $v(\mathbf{X})$ is derived with the aid of mixed μ . We begin the derivation by ‘‘pulling out’’ $\hat{\mathbf{U}}(w)$ from $\hat{\Phi}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{\Lambda})$ and considering the composite system of $\hat{\mathbf{P}}_g(w)$, \mathbf{X} and $\mathbf{\Lambda}$. The description of this composite system can be obtained as

$$\begin{cases} \mathbf{x}_{pc}(k+1) = (\bar{\mathbf{A}}(\mathbf{X}) + \mathbf{B}_u \mathbf{\Lambda} \mathbf{C}_u) \mathbf{x}_{pc}(k) + \mathbf{B}_v \mathbf{v}(k) \\ \quad + \bar{\mathbf{B}}(\mathbf{X}) \mathbf{w}(k), \\ \mathbf{h}(k) = \bar{\mathbf{C}}_h \mathbf{x}_{pc}(k) + \mathbf{D}_{1,1} \mathbf{v}(k) + \mathbf{D}_{1,2} \mathbf{w}(k), \\ \mathbf{z}(k) = \bar{\mathbf{C}}(\mathbf{X}) \mathbf{x}_{pc}(k) + \mathbf{D}_{2,1} \mathbf{v}(k) + \bar{\mathbf{D}}(\mathbf{X}) \mathbf{w}(k), \end{cases} \quad (22)$$

where

$$\bar{\mathbf{A}}(\mathbf{X}) = \begin{bmatrix} \mathbf{A}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \mathbf{X} \begin{bmatrix} \mathbf{C}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \triangleq \mathbf{M}_0 + \mathbf{M}_1 \mathbf{X} \mathbf{M}_2 \in \mathcal{R}^{(n+m) \times (n+m)}, \quad (23)$$

$$\mathbf{B}_u \triangleq \mathbf{d}_{t+m} \otimes \mathbf{M}_1 \in \mathcal{R}^{(n+m) \times N}, \quad (24)$$

$$\mathbf{C}_u \triangleq \mathbf{M}_2 \otimes \mathbf{d}_{s+m}^T \in \mathcal{R}^{N \times (n+m)}, \quad (25)$$

$$\mathbf{B}_{\bar{v}} = [\mathbf{B}_{\bar{v}}^T \quad \mathbf{0}]^T \in \mathcal{R}^{(n+m) \times n_1}, \quad (26)$$

$$\bar{\mathbf{B}}(\mathbf{X}) = [\mathbf{B}_{\bar{v}}^T \quad \mathbf{0}]^T + \mathbf{M}_1 \mathbf{X} [\mathbf{D}_{3,2}^T \quad \mathbf{0}]^T \triangleq \mathbf{B}_{\bar{w}} + \mathbf{M}_1 \mathbf{X} \mathbf{N}_2 \in \mathcal{R}^{(n+m) \times n_2}, \quad (27)$$

$$\mathbf{C}_{\bar{h}} = [\mathbf{C}_{\bar{h}} \quad \mathbf{0}] \in \mathcal{R}^{n_1 \times (n+m)}, \quad (28)$$

$$\bar{\mathbf{C}}(\mathbf{X}) = [\mathbf{C}_{\bar{z}} \quad \mathbf{0}] + [\mathbf{D}_{2,3} \quad \mathbf{0}] \mathbf{X} \mathbf{M}_2 \triangleq \mathbf{C}_{\bar{z}} + \mathbf{N}_1 \mathbf{X} \mathbf{M}_2 \in \mathcal{R}^{n_2 \times (n+m)}, \quad (29)$$

$$\bar{\mathbf{D}}(\mathbf{X}) = \mathbf{D}_{2,2} + \mathbf{N}_1 \mathbf{X} \mathbf{N}_2 \in \mathcal{R}^{n_2 \times n_2}, \quad (30)$$

$$\mathbf{x}_{PC}(k) \triangleq [\mathbf{x}_p^T(k) \quad \mathbf{x}_c^T(k)]^T. \quad (31)$$

The transfer function matrix of (22) is

$$\hat{\Psi}(w, \mathbf{X}, \Lambda) \triangleq w \begin{bmatrix} \mathbf{C}_{\bar{h}} \\ \bar{\mathbf{C}}(\mathbf{X}) \end{bmatrix} (\mathbf{I} - w(\bar{\mathbf{A}}(\mathbf{X}) + \mathbf{B}_u \Lambda \mathbf{C}_u))^{-1} \times [\mathbf{B}_{\bar{v}} \quad \bar{\mathbf{B}}(\mathbf{X})] + \begin{bmatrix} \mathbf{D}_{1,1} & \mathbf{D}_{1,2} \\ \mathbf{D}_{2,1} & \bar{\mathbf{D}}(\mathbf{X}) \end{bmatrix}, \quad (32)$$

where $\hat{\Psi}(w, \mathbf{X}, \Lambda) \in \mathcal{C}^{(n_1+n_2) \times (n_1+n_2)}$ for any $w \in \mathcal{U}$. Let

$$\mathcal{K}_{\psi} \triangleq \left\{ \mathbf{Y}_{\psi} = \begin{bmatrix} \mathbf{Y}_{\psi} = \text{diag}(\theta_1 \mathbf{I}_{p_1}, \dots, \theta_b \mathbf{I}_{p_b}, \\ \boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_{d+1}) \in \mathcal{C}^{(n_1+n_2) \times (n_1+n_2)} \\ \forall i \in \{1, \dots, b\}, \theta_i \in \mathcal{C} \\ \forall j \in \{1, \dots, d\}, \boldsymbol{\Omega}_j \in \mathcal{C}^{p_{b+j} \times p_{b+j}} \\ \boldsymbol{\Omega}_{d+1} \in \mathcal{C}^{n_2 \times n_2} \end{bmatrix} \right\}.$$

Then, we can obtain the corresponding $\mu_{\mathcal{K}_{\psi}}(\hat{\Psi}(w, \mathbf{X}, \Lambda))$. The following result on robust performance (Zhou, Doyle, & Glover, 1996) links $\nu(\mathbf{X})$ to $\mu_{\mathcal{K}_{\psi}}(\hat{\Psi}(w, \mathbf{X}, \Lambda))$.

Lemma 4. For $\mathbf{X} \in \mathcal{R}^{(s+m) \times (t+m)}$, if there exists $0 \leq \beta \in \mathcal{R}$ such that

$$\hat{\Psi}(w, \mathbf{X}, \Lambda) \text{ is stable, } \forall \Lambda \in \mathcal{O}_{\beta}, \quad (33)$$

$$\left\{ \begin{array}{l} \mu_{\mathcal{K}_{\psi}} \left(\text{diag} \left(\tau \mathbf{I}_{n_1}, \frac{1}{\xi} \mathbf{I}_{n_2} \right) \hat{\Psi}(w, \mathbf{X}, \Lambda) \right) < 1, \\ \forall w \in \mathcal{U}, \forall \Lambda \in \mathcal{O}_{\beta}, \end{array} \right. \quad (34)$$

then $\mathbf{X} \in \mathcal{X}_m$ and $\beta < \nu(\mathbf{X})$.

The problem in dealing with (33) and (34) is that $\hat{\Psi}(w, \mathbf{X}, \Lambda)$ contains an indeterminate w and Λ . For this reason, we need the following theorem.

Theorem 1. For $\mathbf{X} \in \mathcal{R}^{(s+m) \times (t+m)}$, if there exists $0 \leq \beta \in \mathcal{R}$ such that

$$\mu_{\mathcal{K}_{\theta}}(\Theta(\mathbf{X}, \beta)) < 1, \quad (35)$$

then (33) and (34) hold. In (35),

$$\Theta(\mathbf{X}, \beta) \triangleq \begin{bmatrix} \bar{\mathbf{A}}(\mathbf{X}) & \mathbf{B}_u & \mathbf{B}_{\bar{v}} & \bar{\mathbf{B}}(\mathbf{X}) \\ \beta \mathbf{C}_u & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \tau \mathbf{C}_{\bar{h}} & \mathbf{0} & \tau \mathbf{D}_{1,1} & \tau \mathbf{D}_{1,2} \\ \frac{1}{\xi} \bar{\mathbf{C}}(\mathbf{X}) & \mathbf{0} & \frac{1}{\xi} \mathbf{D}_{2,1} & \frac{1}{\xi} \bar{\mathbf{D}}(\mathbf{X}) \end{bmatrix}, \quad (36)$$

$$\mathcal{K}_{\theta} \triangleq \{ \text{diag}(\mathbf{Y}_h, \mathbf{Y}_{\psi}) \mid \mathbf{Y}_h \in \mathcal{K}_h, \mathbf{Y}_{\psi} \in \mathcal{K}_{\psi} \}, \quad (37)$$

$$\mathcal{K}_h \triangleq \{ \mathbf{Y}_h = \text{diag}(w \mathbf{I}_{n+m}, \Lambda) \mid w \in \mathcal{C}, \Lambda \in \mathcal{O} \}. \quad (38)$$

Proof. Denote

$$\mathbf{H}(\mathbf{X}, \beta) \triangleq \begin{bmatrix} \bar{\mathbf{A}}(\mathbf{X}) & \mathbf{B}_u \\ \beta \mathbf{C}_u & \mathbf{0} \mathbf{I}_N \end{bmatrix}.$$

By Lemma 3, (35) is equivalent to

$$\mu_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X}, \beta)) < 1, \quad (39)$$

$$\mu_{\mathcal{K}_{\psi}}(F(\Theta(\mathbf{X}, \beta), \mathbf{Y}_h)) < 1, \quad \forall \mathbf{Y}_h \in \mathcal{BK}_h, \quad (40)$$

where $\mathcal{BK}_h \triangleq \{ \mathbf{Y}_h \mid \mathbf{Y}_h \in \mathcal{K}_h, \bar{\sigma}(\mathbf{Y}_h) \leq 1 \}$. Define

$$\mathcal{K}_a \triangleq \{ w \mathbf{I}_{n+m} \mid w \in \mathcal{C} \}, \quad \mathcal{K}_0 \triangleq \mathcal{O} \quad (41)$$

which are compatible with $\bar{\mathbf{A}}(\mathbf{X})$ and $\mathbf{0} \mathbf{I}_N$, respectively. Since $\bar{\mathbf{A}}(\mathbf{X})$ is stable and \mathcal{K}_a contains perturbations of one repeated complex scalar, we conclude by Lemma 1 that

$$\mu_{\mathcal{K}_a}(\bar{\mathbf{A}}(\mathbf{X})) = \rho(\bar{\mathbf{A}}(\mathbf{X})) < 1. \quad (42)$$

Thus, again from Lemma 3, (39) means that $\forall w \in \mathcal{U}$,

$$\begin{aligned} \mu_{\mathcal{K}_0}(F(\mathbf{H}(\mathbf{X}, \beta), w \mathbf{I}_{n+m})) \\ = \mu_{\mathcal{K}_0}(\beta \mathbf{C}_u (\mathbf{I} - w \bar{\mathbf{A}}(\mathbf{X}))^{-1} w \mathbf{B}_u) < 1. \end{aligned} \quad (43)$$

It is known from the stability of $\bar{\mathbf{A}}(\mathbf{X})$ that $\mathbf{I} - w \bar{\mathbf{A}}(\mathbf{X})$ is invertible for any $w \in \mathcal{U}$. Then, (43) and (6) imply

$$\begin{aligned} \inf_{\substack{\Lambda \in \mathcal{O} \\ w \in \mathcal{U}}} \{ \bar{\sigma}(\Lambda) \mid \det(\mathbf{I} - w(\mathbf{I} - w \bar{\mathbf{A}}(\mathbf{X}))^{-1} \mathbf{B}_u \Lambda \mathbf{C}_u) = 0 \} \\ = \inf_{\substack{\Lambda \in \mathcal{O} \\ w \in \mathcal{U}}} \{ \bar{\sigma}(\Lambda) \mid \det(\mathbf{I} - w \bar{\mathbf{A}}(\mathbf{X}) - w \mathbf{B}_u \Lambda \mathbf{C}_u) = 0 \} \\ = \inf_{\Lambda \in \mathcal{O}} \{ \bar{\sigma}(\Lambda) \mid \bar{\mathbf{A}}(\mathbf{X}) + \mathbf{B}_u \Lambda \mathbf{C}_u \text{ is unstable} \} > \beta. \end{aligned}$$

Thus, (33) holds. For any $\mathbf{Y}_h = \text{diag}(w \mathbf{I}_{n+m}, \Lambda_0) \in \mathcal{BK}_h$,

$$\begin{aligned} F(\Theta(\mathbf{X}, \beta), \mathbf{Y}_h) &= \begin{bmatrix} \tau \mathbf{D}_{1,1} & \tau \mathbf{D}_{1,2} \\ \frac{1}{\xi} \mathbf{D}_{2,1} & \frac{1}{\xi} \bar{\mathbf{D}}(\mathbf{X}) \end{bmatrix} + \begin{bmatrix} \tau \mathbf{C}_{\bar{h}} & \mathbf{0} \\ \frac{1}{\xi} \bar{\mathbf{C}}(\mathbf{X}) & \mathbf{0} \end{bmatrix} \\ &\times (\mathbf{I} - \mathbf{Y}_h \mathbf{H}(\mathbf{X}, \beta))^{-1} \mathbf{Y}_h \begin{bmatrix} \mathbf{B}_{\bar{v}} & \bar{\mathbf{B}}(\mathbf{X}) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \tau \mathbf{D}_{1,1} & \tau \mathbf{D}_{1,2} \\ \frac{1}{\xi} \mathbf{D}_{2,1} & \frac{1}{\xi} \bar{\mathbf{D}}(\mathbf{X}) \end{bmatrix} + w \begin{bmatrix} \tau \mathbf{C}_{\bar{h}} \\ \frac{1}{\xi} \bar{\mathbf{C}}(\mathbf{X}) \end{bmatrix} \\ &\times (\mathbf{I} - w(\bar{\mathbf{A}}(\mathbf{X}) + \beta \mathbf{B}_u \Lambda_0 \mathbf{C}_u))^{-1} [\mathbf{B}_{\bar{v}} \quad \bar{\mathbf{B}}(\mathbf{X})] \\ &= \text{diag} \left(\tau \mathbf{I}_{n_1}, \frac{1}{\xi} \mathbf{I}_{n_2} \right) \hat{\Psi}(w, \mathbf{X}, \beta \Lambda_0). \end{aligned} \quad (44)$$

Thus, (40) guarantees that (34) holds. \square

Due to the well-known difficulty in computing the value of $\mu_{\mathcal{K}_{\theta}}(\Theta(\mathbf{X}, \beta))$, we replace $\mu_{\mathcal{K}_{\theta}}(\Theta(\mathbf{X}, \beta))$ with $\alpha_{\mathcal{K}_{\theta}}(\Theta(\mathbf{X}, \beta))$.

Corollary 2. For $\mathbf{X} \in \mathcal{R}^{(s+m) \times (t+m)}$, if there exists $0 \leq \beta \in \mathcal{R}$ such that $\alpha_{\mathcal{K}_{\theta}}(\Theta(\mathbf{X}, \beta)) < 1$, then $\mathbf{X} \in \mathcal{X}_m$ and $\beta < \nu(\mathbf{X})$.

Based on Corollary 2, define

$$\tilde{\mathcal{X}}_m \triangleq \{ \mathbf{X} \mid \mathbf{X} \in \mathcal{R}^{(s+m) \times (t+m)}, \alpha_{\mathcal{K}_{\theta}}(\Theta(\mathbf{X}, 0)) < 1 \}, \quad (45)$$

which obviously is a subset of \mathcal{X}_m . For $\mathbf{X} \in \tilde{\mathcal{X}}_m$, define

$$\tilde{\nu}(\mathbf{X}) \triangleq \sup_{0 \leq \beta \in \mathcal{R}} \{ \beta \mid \alpha_{\mathcal{K}_{\theta}}(\Theta(\mathbf{X}, \beta)) < 1 \}, \quad (46)$$

which obviously is a lower bound of $v(\mathbf{X})$ and can also be viewed as an FWL performance measure. For \mathcal{K}_θ given in (37), the related positive definite matrix set

$$\mathcal{E}_{\mathcal{R}\mathcal{K}_\theta} \triangleq \left\{ \mathbf{E} \begin{cases} \mathbf{E} = \text{diag}(\mathbf{E}_1, e_1, \dots, e_N, \mathbf{E}_2, \dots, \mathbf{E}_{b+1}), \\ \eta_1 \mathbf{I}_{p_{b+1}}, \dots, \eta_d \mathbf{I}_{p_{b+d}}, \eta_{d+1} \mathbf{I}_{n_2} \\ \mathbf{0} < \mathbf{E}_1 \in \mathcal{R}^{(n+m) \times (n+m)} \\ \forall i \in \{1, \dots, b\}, 0 < \mathbf{E}_{i+1} \in \mathcal{R}^{p_i \times p_i} \\ 0 < e_1, \dots, e_N, \eta_1, \dots, \eta_{d+1} \in \mathcal{R} \end{cases} \right\}$$

is defined. It is interesting to see that $\Theta(\mathbf{X}, \beta)$ and \mathcal{K}_θ satisfy the condition of Corollary 1 and hence $\tilde{v}(\mathbf{X})$ is computable by solving the following optimisation problem

$$\tilde{v}(\mathbf{X}) = \sup_{0 \leq \beta \in \mathcal{R}} \beta, \tag{47}$$

$$\text{s.t. } \begin{cases} \mathbf{E} > \Theta^T(\mathbf{X}, \beta) \mathbf{E} \Theta(\mathbf{X}, \beta), \\ \mathbf{E} \in \mathcal{E}_{\mathcal{R}\mathcal{K}_\theta}, \end{cases}$$

based on the combined Linear Matrix Inequality (LMI) technique and bisection search.

4. Robust FWL controller design

With the tractable FWL performance measure $\tilde{v}(\mathbf{X})$, the proposed robust FWL controller design problem can now be summarised. Given $\hat{\mathbf{P}}_g(w)$, τ , ξ , m and assuming a nonempty $\tilde{\mathcal{X}}_m$, find a controller $\mathbf{X}_{\text{opt}} \in \tilde{\mathcal{X}}_m$ that achieves

$$\gamma = \sup_{\mathbf{X} \in \tilde{\mathcal{X}}_m} \tilde{v}(\mathbf{X}). \tag{48}$$

It is seen easily that our design objective is to make the FWL tolerance as large as possible, while satisfying a suboptimal robust control requirement. A large FWL tolerance means that the resulting controller can be implemented with a processor of short word length. Combining (23), (27), (29), (30), (36), (47) and (48), the above design problem can also be expressed as

$$\gamma = \sup_{0 \leq \beta \in \mathcal{R}} \beta, \tag{49}$$

$$\text{s.t. } \begin{cases} \mathbf{E} > (\mathbf{Y}_\beta + \mathbf{Y}_1 \mathbf{X} \mathbf{Y}_2)^T \mathbf{E} (\mathbf{Y}_\beta + \mathbf{Y}_1 \mathbf{X} \mathbf{Y}_2), \\ \mathbf{E} \in \mathcal{E}_{\mathcal{R}\mathcal{K}_\theta}, \\ \mathbf{X} \in \mathcal{R}^{(s+m) \times (t+m)}, \end{cases}$$

where

$$\mathbf{Y}_\beta \triangleq \begin{bmatrix} \mathbf{M}_0 & \mathbf{B}_u & \mathbf{B}_{\bar{v}} & \mathbf{B}_{\bar{w}} \\ \beta \mathbf{C}_u & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \tau \mathbf{C}_{\bar{h}} & \mathbf{0} & \tau \mathbf{D}_{1,1} & \tau \mathbf{D}_{1,2} \\ \frac{1}{\xi} \mathbf{C}_{\bar{z}} & \mathbf{0} & \frac{1}{\xi} \mathbf{D}_{2,1} & \frac{1}{\xi} \mathbf{D}_{2,2} \end{bmatrix}, \tag{50}$$

$$\mathbf{Y}_1 \triangleq \begin{bmatrix} \mathbf{M}_1^T & \mathbf{0} \\ \frac{1}{\xi} \mathbf{N}_1^T \end{bmatrix}^T \in \mathcal{R}^{(n+m+N+n_1+n_2) \times (s+m)}, \tag{51}$$

$$\mathbf{Y}_2 \triangleq \begin{bmatrix} \mathbf{M}_2 & \mathbf{0} & \mathbf{N}_2 \end{bmatrix} \in \mathcal{R}^{(t+m) \times (n+m+N+n_1+n_2)}. \tag{52}$$

The optimisation problem (49) contains a Bilinear Matrix Inequality (BMI) (Kanev, Scherer, Verhaegen, & De Schutter, 2004; VanAntwerp & Braatz, 2000) whose size is $2(n+m+N+n_1+n_2)$. In the work Yang et al. (2001), an FWL H_2 controller design problem is studied which is also formulated as a BMI optimisation problem. However, as the method proposed in Yang et al. (2001) examines each vertice of the FWL perturbation hypercube \mathcal{D}_β , the related optimisation problem contains at least 2^N BMIs of a size no less than $4n$. Considering a design example with $n = 6$, $m = 2$, $s = 1$, $t = 1$, $n_1 = 1$ and $n_2 = 1$, for instance, the optimisation problem formulated in this paper has only one BMI of size 38, while the corresponding optimisation problem proposed in Yang et al. (2001) has

over 512 BMIs of size 24. Undoubtedly, our proposed method has significant computational advantages over the method proposed in Yang et al. (2001).

Next, we discuss how to solve the BMI problem (49). The following result (Iwasaki, 1993, 1999) is useful in solving the nonconvex optimisation problem (49).

Lemma 5. Suppose that $\mathbf{Y}_1^T \mathbf{Y}_1 > 0$ and $\mathbf{Y}_2 \mathbf{Y}_2^T > 0$. Give a $0 < \omega \in \mathcal{R}$ and a $0 \leq \beta \in \mathcal{R}$. If and only if there exist $0 < \mathbf{E} \in \mathcal{E}_{\mathcal{R}\mathcal{K}_\theta}$, $\mathbf{J} \in \mathcal{R}^{(s+m) \times (n+m+N+n_1+n_2)}$ and $\mathbf{L} \in \mathcal{R}^{(n+m+N+n_1+n_2) \times (t+m)}$ such that

$$\begin{cases} \omega \mathbf{E} > (\mathbf{Y}_\beta + \mathbf{Y}_1 \mathbf{J})^T \mathbf{E} (\mathbf{Y}_\beta + \mathbf{Y}_1 \mathbf{J}), \\ \omega \mathbf{E} > (\mathbf{Y}_\beta + \mathbf{L} \mathbf{Y}_2)^T \mathbf{E} (\mathbf{Y}_\beta + \mathbf{L} \mathbf{Y}_2), \end{cases} \tag{53}$$

then there exists $\mathbf{X} \in \mathcal{R}^{(s+m) \times (t+m)}$ such that

$$\omega \mathbf{E} > (\mathbf{Y}_\beta + \mathbf{Y}_1 \mathbf{X} \mathbf{Y}_2)^T \mathbf{E} (\mathbf{Y}_\beta + \mathbf{Y}_1 \mathbf{X} \mathbf{Y}_2). \tag{54}$$

The above lemma shows that (54) can be transformed into (53). It is easy to see that (53) actually is an LMI when \mathbf{J} is given. Moreover, (53) is equivalent to

$$\begin{cases} \omega \mathbf{E}^{-1} > (\mathbf{Y}_\beta + \mathbf{Y}_1 \mathbf{J}) \mathbf{E}^{-1} (\mathbf{Y}_\beta + \mathbf{Y}_1 \mathbf{J})^T, \\ \omega \mathbf{E}^{-1} > (\mathbf{Y}_\beta + \mathbf{L} \mathbf{Y}_2) \mathbf{E}^{-1} (\mathbf{Y}_\beta + \mathbf{L} \mathbf{Y}_2)^T. \end{cases} \tag{55}$$

The inequality (55) is also an LMI when \mathbf{L} is given. Based on the equivalence relations among (53)–(55), the optimisation problem (49) is solved in this paper using a two-stage procedure. The task of the stage one is to obtain an $\mathbf{L}_{in} \in \mathcal{R}^{(n+m+N+n_1+n_2) \times (t+m)}$ which satisfies

$$\begin{cases} \mathbf{E}^{-1} > (\mathbf{Y}_0 + \mathbf{Y}_1 \mathbf{J}) \mathbf{E}^{-1} (\mathbf{Y}_0 + \mathbf{Y}_1 \mathbf{J})^T, \\ \mathbf{E}^{-1} > (\mathbf{Y}_0 + \mathbf{L}_{in} \mathbf{Y}_2) \mathbf{E}^{-1} (\mathbf{Y}_0 + \mathbf{L}_{in} \mathbf{Y}_2)^T, \end{cases} \tag{56}$$

for some $0 < \mathbf{E} \in \mathcal{E}_{\mathcal{R}\mathcal{K}_\theta}$ and $\mathbf{J} \in \mathcal{R}^{(s+m) \times (n+m+N+n_1+n_2)}$, where \mathbf{Y}_0 is the value of \mathbf{Y}_β at $\beta = 0$. In stage two, the problem (49) is solved with the feasible starting point \mathbf{L}_{in} . The details of the stage-two algorithm are as follows.

Step (1) Let the iterative index be $i = 0$ and $\mathbf{L}_{(i)} = \mathbf{L}_{in}$, and set N_{it} to a sufficiently large integer.

Step (2) Solve

$$\sup_{0 \leq \beta \in \mathcal{R}} \beta, \tag{57}$$

$$\text{s.t. } \begin{cases} \mathbf{E}^{-1} > (\mathbf{Y}_\beta + \mathbf{Y}_1 \mathbf{J}) \mathbf{E}^{-1} (\mathbf{Y}_\beta + \mathbf{Y}_1 \mathbf{J})^T, \\ \mathbf{E}^{-1} > (\mathbf{Y}_\beta + \mathbf{L}_{(i)} \mathbf{Y}_2) \mathbf{E}^{-1} (\mathbf{Y}_\beta + \mathbf{L}_{(i)} \mathbf{Y}_2)^T, \\ 0 < \mathbf{E} \in \mathcal{E}_{\mathcal{R}\mathcal{K}_\theta}, \\ \mathbf{J} \in \mathcal{R}^{(s+m) \times (n+m+N+n_1+n_2)}, \end{cases}$$

by the combined LMI technique and bisection search. Let a maximiser be $\mathbf{J}_{(i)}$.

Step (3) Solve

$$\beta_{i+1} = \sup_{0 \leq \beta \in \mathcal{R}} \beta, \tag{58}$$

$$\text{s.t. } \begin{cases} \mathbf{E} > (\mathbf{Y}_\beta + \mathbf{Y}_1 \mathbf{J}_{(i)})^T \mathbf{E} (\mathbf{Y}_\beta + \mathbf{Y}_1 \mathbf{J}_{(i)}), \\ \mathbf{E} > (\mathbf{Y}_\beta + \mathbf{L} \mathbf{Y}_2)^T \mathbf{E} (\mathbf{Y}_\beta + \mathbf{L} \mathbf{Y}_2), \\ 0 < \mathbf{E} \in \mathcal{E}_{\mathcal{R}\mathcal{K}_\theta}, \\ \mathbf{L} \in \mathcal{R}^{(n+m+N+n_1+n_2) \times (t+m)}, \end{cases}$$

by the combined LMI technique and bisection search. Let a maximiser be $\mathbf{L}_{(i+1)}$, and denote $\mathbf{E}_{(i+1)}$ the corresponding positive definite matrix.

Step (4) Set $i = i + 1$. If $i < N_{it}$, go to Step(2); if $i \geq N_{it}$, go to Step(5).

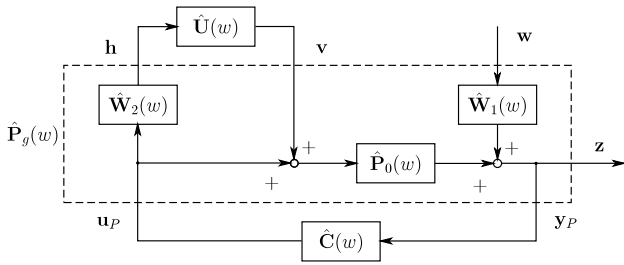


Fig. 1. System configuration of a robust finite-word-length control system design example.

Step (5) Set $\mathbf{E}_d = \mathbf{E}_{(i)}$, denote \mathbf{Y}_d the value of \mathbf{Y}_β at $\beta = \beta_i$, and calculate the optimal controller \mathbf{X}_{opt} through solving (54) with $\omega = 1$, $\mathbf{E} = \mathbf{E}_d$ and $\mathbf{Y}_\beta = \mathbf{Y}_d$.

The algorithm in stage two can be modified to be used in stage one, where the problem

$$\begin{aligned} & \inf_{\omega \in \mathcal{R}} \omega, \\ & \text{s.t. } \omega \mathbf{E} > (\mathbf{Y}_0 + \mathbf{Y}_1 \mathbf{X} \mathbf{Y}_2)^\top \mathbf{E} (\mathbf{Y}_0 + \mathbf{Y}_1 \mathbf{X} \mathbf{Y}_2), \\ & \quad 0 < \mathbf{E} \in \mathcal{E}_{\mathcal{R} \times \mathcal{X}_q}, \\ & \quad \mathbf{X} \in \mathcal{R}^{(s+m) \times (t+m)}, \end{aligned} \tag{59}$$

is solved until $\omega < 1$.

5. Numerical design examples

The first example is shown in Fig. 1, where

$$\begin{aligned} \hat{\mathbf{P}}_0(w) &= \frac{3.3750 \times 10^{-3}w + 1.3669 \times 10^{-2}w^2 + 3.4605 \times 10^{-3}w^3}{1 - 3.0488w + 3.1001w^2 - 1.0513w^3}, \\ \hat{\mathbf{W}}_1(w) &= \frac{4.9875 \times 10^{-3}w}{1 - 9.9501 \times 10^{-1}w}, \\ \hat{\mathbf{W}}_2(w) &= \frac{5.8512 \times 10^{-1}w - 5.5933 \times 10^{-1}w^2}{1 - 1.3390w + 3.7908 \times 10^{-1}w^2}, \end{aligned}$$

and the plant model uncertainty $\hat{\mathbf{U}}(w) \in \mathcal{H}_\tau$ with $\tau = 0.4$. From the given $\hat{\mathbf{P}}_0(w)$, $\hat{\mathbf{W}}_1(w)$ and $\hat{\mathbf{W}}_2(w)$, it was easy to obtain the nominal plant model $\hat{\mathbf{P}}_g(w)$. For this example, the constant ξ that bounds the closed-loop H_∞ norm from \mathbf{w} to \mathbf{z} was set to $\xi = 0.3$, and the controller order was chosen to be $m = 2$. The task was thus to design a 2nd-order controller based on the robust FWL performance measure \tilde{v} .

For this design example, the optimisation problem (49) was formulated and the algorithm described in Section 4 was used to find solutions of the optimal robust FWL design problem (49). The resulting controller was

$$\mathbf{X}_{\text{opt1}} = \begin{bmatrix} -103.44 & -15.600 & -1.4984 \\ -16.070 & -1.4261 & 0.25055 \\ -19.469 & -3.0400 & 0.37517 \end{bmatrix}$$

achieving $\tilde{v}(\mathbf{X}_{\text{opt1}}) = 8.2842 \times 10^{-3}$. This designed controller achieves the required robust control performance and is also robust to FWL perturbation errors because, for any FWL perturbation to \mathbf{X}_{opt1} smaller than 8.2842×10^{-3} and for any $\hat{\mathbf{U}}(w) \in \mathcal{H}_\tau$ with $\tau = 0.4$, the closed-loop system maintains stability and the closed-loop H_∞ norm from \mathbf{w} to \mathbf{z} is always less than 0.3.

Using a fixed point processor of c -bit length to implement a realization \mathbf{X} , we can assign the c bits as: 1 bit for the sign, c_{int} bits for the integer part, and c_{fra} bits for the fraction part. To provide a sufficient dynamic range for \mathbf{X} , at least $c_{\text{int}} = \lceil \log_2 \|\mathbf{X}\|_m \rceil$, where $\lceil x \rceil$ denotes the closest integer greater than or equal to $x \in \mathcal{R}$. The

fraction bit length bounds the absolute values of the FWL errors by $2^{-(c_{\text{fra}}+1)}$. Comparing this bound with the measure $\tilde{v}(\mathbf{X})$ within which the closed-loop performance is maintained, it is known that at least $c_{\text{fra}} = \lceil -\log_2 \tilde{v}(\mathbf{X}) \rceil - 1$. Therefore, when implementing \mathbf{X} with fixed point processor, $\tilde{c}(\mathbf{X}) \triangleq \lceil \log_2 \|\mathbf{X}\|_m \rceil + \lceil -\log_2 \tilde{v}(\mathbf{X}) \rceil$ can be viewed as the minimal word length guaranteeing closed-loop performance, estimated based on $\tilde{v}(\mathbf{X})$. In this example, $\tilde{c}(\mathbf{X}_{\text{opt1}}) = 14$.

The second example was from Yang et al. (2001). The original example in Yang et al. (2001) was for the FWL H_2 control under plant parameter uncertainty. Since $\|\hat{\Phi}_{wz}\|_\infty \geq \|\hat{\Phi}_{wz}\|_2$ and structured uncertainty includes parameter uncertainty, we substituted $\|\hat{\Phi}_{wz}\|_\infty$ for $\|\hat{\Phi}_{wz}\|_2$ and substituted plant structured uncertainty for plant parameter uncertainty to obtain our problem formulation. Thus the second example was provided as

$$\begin{aligned} \mathbf{A}_p &= \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0 \end{bmatrix}, & \mathbf{B}_v &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \mathbf{B}_w &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \\ \mathbf{B}_p &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \mathbf{C}_h &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, & \mathbf{C}_z &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \\ \mathbf{D}_{2,2} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \mathbf{D}_{2,3} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & \mathbf{C}_p &= [0 \quad -1], \\ \mathbf{D}_{3,2} &= [1 \quad 1], & \mathbf{D}_{1,1} = \mathbf{D}_{1,2} = \mathbf{D}_{2,1} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \hat{\mathbf{U}}(w) &= \varphi(w) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{H}_\tau \text{ with } \varphi(w) \in \mathcal{C}, \text{ and } \tau = 0.13. \end{aligned}$$

Set the constant $\xi = 4.9676$. We designed a 1st-order controller by minimising \tilde{v} . The resulting controller was

$$\mathbf{X}_{\text{opt2}} = \begin{bmatrix} 1.0853 & -0.36600 \\ 1.1031 & -0.34734 \end{bmatrix}$$

with $\tilde{v}(\mathbf{X}_{\text{opt2}}) = 0.0275$, which can be implemented with a processor of $\tilde{c}(\mathbf{X}_{\text{opt2}}) = 7$ bits. As $\|\hat{\Phi}_{wz}\|_\infty \geq \|\hat{\Phi}_{wz}\|_2$, the system was guaranteed to be closed-loop stable and $\|\hat{\Phi}_{wz}\|_2 < 4.9676$ when $\tau = 0.13$ and the FWL bound was 0.0275. By direct optimising $\|\hat{\Phi}_{wz}\|_2$ under uncertainty, Yang et al. (2001) obtained a controller achieving $\|\hat{\Phi}_{wz}\|_2 < 3.0822$ when $\tau = 0.13$ and the FWL bound 0.0275. For this example, 32 BMIs of size 8 were solved in Yang et al. (2001) while one BMI of size 22 was solved using our method.

6. Conclusions

A robust FWL controller design approach has been proposed based on the mixed μ theory. We have defined a novel FWL robust control performance measure which takes into account both the standard robust control requirements and the FWL implementation considerations. This FWL robust control performance measure can be computed conveniently using an LMI method. The corresponding optimal FWL robust controller design problem has been formulated naturally as a mixed μ problem which can be solved by means of BMI techniques.

As mentioned in Section 1, this paper investigates only the FWL parameter representation errors while in fact the FWL arithmetic rounding errors also occur in the digital controller. Therefore, after obtaining \mathbf{X}_{opt} and the corresponding word length $\tilde{c}(\mathbf{X}_{\text{opt}})$, the effect of rounding errors must be tested and we suggest to simulate the designed closed-loop system with a range of the expected working signals. In the simulation, the controller \mathbf{X}_{opt} is first implemented with $\tilde{c}(\mathbf{X}_{\text{opt}})$ bits of fixed-point representation and arithmetic. If there are no bounded limit cycles or unbounded response occurring in the simulation process, the FWL rounding

errors will not lead to instability (Miller, Mousa, & Michel, 1988) and hence $\tilde{c}(X_{opt})$ is sufficient. If bounded limit cycles or unbounded response are observed, the bit length should be increased until they disappear. Our future research will study how to design robust controllers that simultaneously consider both the two types of FWL errors.

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